TEACHING OF MATHEMATICS
IN
SECONDARY SCHOOLS

S. K. CHAKRABARTY

WEST BENGAL BOARD OF SECONDARY EDUCATION
77/2, Park Street, Calcutta-16

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By

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FOREWORD

That Mathematics has an important place to occupy in the scheme of studies for general education, is disputed by few. Keeping in view the importance of the study of Mathematics and its bearing on sciences, the West Bengal Board of Secondary Education decided to introduce Mathematics as one of the compulsory subjects for all students in secondary schools from January, 1974.

To what extent mathematics should be taught, or in other words, what should be the course content of school mathematics, is a matter on which opinions may differ. Nevertheless, some beginning has got to be made and the Board expects that teachers and experts in the subject would come forward with suggestions for improvement in the light of their experience.

Certain objectives have been set forth in the reorganised pattern of secondary education effective in the State from the year 1974. Too much emphasis on bookish education should go and education must be related to life. Teaching of mathematics in schools should "enable pupils to solve speedily numerical and geometrical problems that arise in school and family situations and in community activities". Faculty of reasoning has got to be developed and along with should develop pupil's power of application of mathematical skill to problems concerning life and society. With that end in view mathematics syllabus should be modernised.

The Board had to take into account the controversy over introduction of what is called New Mathematics in place of the Traditional one. Axiomatic approach has, no doubt, its own merit. But it is doubtful whether a new set of expressions and symbols alone will make mathematics more useful. I fully agree with the views expressed by Professor S. K. Chakrabarty when he says that "some new mathematics may be useful to them, but that should not be presented as a separate entity and in replacement of some useful traditional topics".

The West Bengal Board of Secondary Education will remain indebted to Dr. S. K. Chakrabarty, for preparing a book entitled "Teaching of Mathematics in Secondary Schools" which may serve as a Guide Book. Professor Chakrabarty has offered valuable suggestions for introducing 'modern concepts' in school mathematics within the framework of the syllabus prepared by the Board. Presentation of mathematics to teen-agers is an important as well as a difficult task. Much depends on how the subject is introduced. I am confident that teachers will find the subject-matter included in the book quite interesting and instructive. Elementary concepts of the Algebra of Sets and Boolean Algebra, and simple concepts of Transformation Geometry coupled with proper understanding of Mathematics for solving problems will, to a great extent, modernise our approach to the subject.

I have great pleasure in commending the publication to teachers and persons interested in making mathematics more useful and meaningful.

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PREFACE

School education in general, must be broad-based and to achieve that objective it is necessary to prescribe a curriculum from the initial to the final stage. The teachers are certainly the competent persons to frame a proper syllabus and also to plan their proper presentation. However, when a large number of students are to be trained on a common pattern a proper curriculum and syllabus for the average students should be presented and some indications as to the mode of presentation of the subject will be helpful to both the subject teachers, and the authors, who may write the textbooks. The quantum of the subject-matter, particularly in mathematics and science, even for the school stage, is increasing with time, but the total time available for presentation cannot be increased proportionately. It is therefore necessary to devise improved methods of presentation and also occasionally weed out superflous materials. Thus the syllabus should be revised frequently and made up-to-date, and some indications as to its proper presentation in the class will be helpful. The present book is an attempt to help the teachers of mathematics and the prospective authors in that respect.

There is a good controversy, in almost all countries of the world, regarding the nature of the material that may be incorporated in the syllabus for school mathematics and whether 'traditional' mathematics should be continued or 'new' mathematics should replace the traditional course. It is difficult to appreciate the attempts of those who want to thrust upon the young pupils of the whole country something new without testing its efficacy or superiority. Experiments are no doubt necessary in order to arrive at the most useful method but that may not be made over the entire student population, which may involve the risk of sacrificing a generation of young aspirants.

Elementary mathematics is essential for everybody but not even one per cent of the student population may go in for a

course of Higher mathematics. In consequence the utilitarian aspects of mathematics should have preference over the more abstract and rigorous fundamentals. The real budding talents in mathematics and science should however, be exposed to the more fundamental aspects of mathematics quite early in their life. Thus some 'new' mathematics may be useful to them but that should not be presented as a separate entity and in replacement of some useful traditional topics. The present syllabus has therefore been framed incorporating some new concepts along with the traditional topics. The teachers will have ample scope for introducing modern concepts and rigour while teaching the traditional materials and even the talents will appreciate the abstract material better when they find its correlation with the traditional topics, and also their applications. It is possible that some of the topics referred to above will be unfamiliar to many teachers. The present book is planned to help such teachers by providing some essential background knowledge on those topics and also by indicating the proper phase and procedure for presenting them to the students. Certainly this book alone will not be sufficient for the purpose, but the teachers will get an idea about the modern approach towards rigorous foundations of elementary mathematics.

In Chapter I, I have discussed the aims and scope of elementary mathematics. In Chapter II, I have given my suggestions as to the order and manner of presentation to the class, of the different topics included in the syllabus. I have also indicated there, how and where some preliminary concepts of 'new' mathematics can be introduced. Some important and elementary concepts of the Algebra of Sets and Boolean Algebra have been given in Appendix I, and some simple concepts of 'Transformation geometry' have been given in Appendix II. It is possible that the teachers will find the subjects interesting and will try to learn more by referring to the various standard books on the subjects now available. In Chapter III, I have given some problems which all students will have to face in their later life. They should therefore have a proper understanding of the methods of solving such problems, which will enable them to contribute, with

confidence, towards the development of the community in which they may live. The authors should also provide some more similar problems in proper positions, so that the teachers and students may have no difficulty in having a good practice on what we may call 'consumer mathematics'. In Appendix III, I have given some historical informations about the life and works of a few ancient mathematicians. It is possible that if these information are properly presented, the young pupils will get some inspiration and also feel proud of their rich heritage.

Mathematics is truly international, and the scope and domain of elementary mathematics is similar in all countries of the world. The lower limit must be the same everywhere, though the upper limit may slightly vary depending on the time available for its presentation and also on the manner of presentation. Naturally therefore some experiments are in progress in all countries for deciding upon the most useful material and its presentation, which however, will depend to a large extent, on the number, calibre and environment of the students concerned. Considering the very large mass of students who are required to follow a uniform pattern, probably the traditional syllabus with occasional injection of 'new' and useful material may be more useful. This again is an experiment, but may not be too risky, and certainly can be modified easily in the light of experience gained within a short period. Comments from experts who might be involved in similar experiments in other regions will help in further improvements towards development and progress of future mathematicians.

In course of preparation for this book, I have drawn liberally, from contemporary literatures and particularly from those mentioned in the Bibliography, the ideas and materials relevant to those included in it. I take this opportunity to acknowledge my indebtedness to all authors of these publications. This book is meant primarily for the teachers and the authors who might be interested in writing suitable textbooks. I shall consider it a privilege to have comments and suggestions from the users of this book, which will help me in further improving the contents of this book later.

It is a pleasure to record my indebtedness to the members of the staff of the K. P. Basu Printing Works, Calcutta, for the utmost care taken by them in printing the book within a short time and in a beautiful form.

The essence of mathematics is creativity and should not end with teaching some techniques in computation with numbers. If this book can help all concerned in the realisation of that aspect, even to a small extent, I shall consider my labour fully rewarded.

December, 1973 S. K. CHAKRABARTY

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CHAPTER I

AIMS AND SCOPE OF TEACHING ELEMENTARY MATHEMATICS

The aims of general education have always been primarily to acquire certain cultural values which may be essential for the progress of civilization. Mathematics has always occupied a prominent place in every educational pattern since the dawn of civilization, because of its power in developing logical thinking, in analyzing properly any given situation and then in coming to definite conclusions which can be substantiated. It has always helped in the proper realization of the fundamental principles underlying the vocational tasks faced by human being and their generalizations leading to useful predictions. It has always stimulated the scientific reasoning which ultimately led to the developments of major theories of physical sciences. It has quite often supplied philosophic thoughts and has also destroyed and shaped religious doctrines. Its present day contribution to modern economic theories has been considerable. It helps in forming the habit of systematically and logically developing any problem to its final stages. It also develops in man the power of concentration.

In spite of these utilities a large group of educated men reject mathematics as an intellectual discipline mainly for the defective manner in which it is occasionally presented to the students at all stages and particularly at the elementary stages. School courses and books have presented mathematics quite often as a series of apparently meaningless technical procedures. Although it is realized that purposeful teaching is necessary in order that the topics taught become functional in the life of the community, complete agreement as to the real objectives in teaching a particular subject or to the general objectives of primary or secondary education, does not exist. In the present century utilitarian aspects of education in general and also of

different subjects are to be established particularly by examining their contributions to daily life. The requirements and practical use of mathematics in the two world wars have directed much attention to the applicational aspetcs of mathematics. On the other hand even from the early days of civilization our endeavour to seek the truth has been the primary incentive towards development of knowledge in different fields and mathematics is no exception to this rule. Thus the principal motivations in the study and developments of mathematics have always been (i) to extend the bounds of knowledge by developing the truth based on logic and (ii) to understand and control the environments. While the first aspect will interest the common intelligentsia, the second aspect is more useful to the technologists for raising the material standard of the human being. The two aspects are complimentary and one cannot be divorced from the other. Mathematics detached from its rich intellectual setting in the culture of our civilization will give undue prominence to its technical aspects and the common man who has little use for that will lose all interest in it. Consequently a subject which is basic and elevating will be neglected and even hated by a group of otherwise highly educated people. A glance through the historical development of Mathematics will show how mathematics has helped to guide twentieth century life and thoughts. There are ample evidences to show how a fundamental concept developed for its philosophical aspects only has even after a century been applied to life situations. George Boole developed his algebra during the first half of the nineteenth century, but the same was used in telephone circuits in the thirties of the present century and is now being used extensively in digital computer circuits. Newton was born in 1642 and before he was twenty-five, he put forward three fundamental concepts relating to (i) spectrum of white light, (ii) basic principles of calculus and (iii) the universal laws of gravitation, without probably realizing the practical or utilitarian aspects of these concepts. His philosophic approach was based on the principle that from clearly verifiable phenomena laws are to be framed that may describe nature's behaviour in the precise language of Mathematics.

We now realize how the concepts of Newton have helped in the development of modern spectroscopy with their applications to many diverse fields *viz.* Chemistry, Biology, Physiology. The laws of gravitation have led to a precise understanding of the motion of heavenly bodies and have made possible, within the last decade, a safe journey to moon. Similar examples may be given to show that cultural as well as philosophical aspects of Mathematics should be encouraged even though their technological aspects may not be immediately visible. Quite often mathematical results may be abstract and dry but that should not take away the incentive towards their developments. It is especially true of Mathematics that, while the creative work is done by individuals, all significant results are the fruition of centuries of sustained works and development by a group of scientists. Intellectual curiosity has always been the guiding light towards fundamental discoveries and the need for the common man both in war and peace, has produced the urge for their applications to technological problems.

Keeping in view the above aspects the aims and objects of Mathematics education may be broadly classed as (i) cultural and (ii) practical, and may be generalized as follows :

(1) To develop the power of logical reasoning and to arrive at definite conclusions, based on primary data, which can be substantiated.
(2) To understand the fundamental concepts and processes of mathematics which will enable pupils to solve speedily the numerical and geometrical problems that arise in their school, family and community activities.
(3) To cultivate the qualities of exactness in expression and performances.
(4) To develop admiration for Mathematics for its precision and capacity for helping man in his adventure and discoveries in various fields.

The realization of the above aims depends critically on the methods a teacher uses in teaching mathematics and also to a

certain extent in the manner of presentation of the subjects in the prescribed text-books. If the student is to develop power of reasoning and preciseness he must always be asked to show that through his tasks. Unless the teacher in course of his teaching shows the applications of the fundamental principles the students may not be able to think in terms of the possible application of his knowledge and also may not be interested in the advanced study and research in his mature days. All developing subjects and particularly Mathematics and science are extending their domain of influence quite rapidly. In consequence, for an integrated course of study from the primary stage to the post-graduate level the total quantum of material and their mode of presentation require constant revision. While the volume of the course content, which should be a function of the time allotted to different stages of education, may increase, improved methods of presentation may help in introducing new material in all stages. Though emphasis on both the cultural aspects as well as utilitarian values of a subject are retained, their proportionate share has been changing from time to time. Again because of the large increase in school population both at the primary stage as well as the secondary stage, consequent on the attempts towards introduction of compulsory primary education envisaged in our constitution, the character of the traditional courses has changed and more emphasis is now laid on utilitarian aspects which will be useful in vocational training and in other practical problems facing the average man. This has to some extent lowered the average standard of knowledge that can be imparted and unless some special provision is made the progress of the really talented students will be affected. For convenience of the average students, several topics or concepts which may prove difficult are omitted, and unless better text-books and improved methods of teaching are introduced the growth of talents will in consequence be stunted. In handling the large number of students and simultaneously maintaining a reasonable standard it is necessary to prescribe detail syllabi and also indicate some standard methods of their presentation. Thus the need for better text-books and more qualified teachers cannot be over-

emphasized. The number of students in the secondary stage who may ultimately be a specialist in a subject or even go for higher education and derive benefit from that, may be only about one per cent or probably much less than that of the total student population at that stage. For administrative reasons it is not possible to teach them separately from the average students, nor is that desirable from psychological standpoint. As a result the brighter students often suffer from lack of progress and the poorer students are unable to maintain the pace required. A partial solution to these difficulties may be obtained by providing text-books for average students and including in that some important advanced topics with suitable indications so that the average students may omit that.

Traditionally at the secondary stage Mathematics as a subject includes Arithmetic, Algebra and Geometry. It is now argued that some other useful topics may be introduced at the school level. It is suggested that some elements of Set-theory, Probability and Statistics, Calculus and even simpler ideas of Applied Mathematics may with advantage be introduced into the school curriculum. The choice of subjects and their priorities should depend on the capabilities of the learner in terms of their individual make-up and background experiences rather than on logical sequence. The abstract nature of pure mathematical thoughts makes it difficult for a pupil at the elementary stage to properly assimilate them and it has quite often been criticised by at least a group of experts. Mathematical reasoning on some form should no doubt be developed even at the early stages of education but the more abstract thinking should be encouraged at a later stage when the pupils may have developed the capacity for abstract thinking. It is probably not difficult to realize that all our knowledge begins with experience, and any fundamental concept gets deep appreciation if its application can be observed or its generality over similar traditional concepts can be presented.

The quantum of any subject that may be introduced at any stage of education must also be correlated to the age of the students. The main emphasis of the educational systems, at

all stages, though evolved out of necessity, was from ancient times on cultural developments and probably more on their philosophic aspects with a view to extend the bounds of knowledge. The preparations for various jobs have come only as a by-product. There has been a persistent demand in recent years, mainly from the administrators, that education must be job oriented. The two aspects are fundamentally different but one cannot be ignored in preference to the other. The needs of a common man particularly in a developing country should naturally have more emphasis. The history of development of science and Mathematics shows how these subjects have developed in order to meet the demands during both war and peace. The developments in science have contributed significantly in raising the standard of living of the common man and have helped them in doing their jobs with higher efficiency. Our experience shows that a particular branch of study gets a rapid development when its results can be applied to human welfare. It is therefore obvious why certain branch of study gets more emphasis when it can be directly applied for the benefit of the common man. Apart from the utilitarian aspects it has quite often been observed that sheer intellectual curiosity also provides an excellent motivation for the highest type of work in Mathematics. Johann Kepler who was born in 1571 became interested in astronomy and during the early years of the 17th century analysed the large mass of very accurate data on the motion of the planets collected by Tycho Brahe. After considerable computational works Kepler was able to formulate (in 1609 the first two laws and in 1619 the third law) the laws of planetary motion. These laws are landmarks in the history of astronomy and mathematics and they provided encouragements to Newton to develop further the principles of modern Celestial Mechanics. These ideas led him to establish the universal law of gravitation. A direct consequence of these results is observed to-day in the successful landing of men on the surface of the moon and also in the existence of 'Skylab' revolving round the earth, providing an excellent opportunity for studying the Physics of the outer space without inter-

ference by the earth's atmosphere. The precise information about 'g' has helped the modern technologists in the exploration for oils and minerals and are thereby helping in the developments of world economy. Thus the scope of the subject should also include problems and phenomena which appear to-day purely cultural or of a philosophic nature. This will develop in the young mind the creative faculties and help him in investigating new lines of attack on problems which cannot be solved by the existing methods.

There has been a persistent demand from certain quarters for the revision of school mathematics syllabi as the traditional syllabus is no longer adequate to give the desirable idea to the young mind about the nature of mathematics he may have to face at the higher level. It is argued that the traditional syllabus contain a large proportion of routine computation and manipulation at the expense of abstract ideas which may give enjoyment and thus develop his taste and regard for mathematics. It has been suggested that the young mind should be exposed to some areas of mathematics which are broadly called *modern* or *new* although most of those concepts are fairly old and were developed in the nineteenth century based on the concepts of *sets, relations, mappings* and *operations*. It is true that a considerable portion of present day mathematics can be developed on the postulates and results of the Set Theory originally put forward by George Cantor (1845—1918) and the Transformation Geometry developed by Felix Klein (1849—1925), but that may not be a sufficient ground for incorporating their basic structures in the school syllabus. It will no doubt be advantageous to the real talents who will go in for advanced courses of Mathematics, if they are exposed to these basic concepts early in their life, but the very large group of students may not appreciate their need and are likely to get frightened and in consequence develop apathy for the subject. Their needs may possibly be better served through the traditional methods. It should certainly be appreciated that the word 'need' should not stipulate only such knowledge or capacities which may be indispensable, but also attainments which may have both utili-

tarian as well as cultural values. There is certainly the necessity for encouraging the study of mathematics for its disciplinary values and may be integrated as an essential part of the school education but the other view of incorporating only those selected courses and methods which can be justified from the standpoint of their direct social and practical values will be a short sighted policy. A careful admixture of the two demands will be ideal.

A group of Mathematicians both in Indian and abroad have been advocating in recent years the urgency of incorporating 'modern mathematics' or 'new mathematics' in the school curriculum even from the very early stage. Their recommendations are based on mainly the following two reasons. They mainly emphasise on the introduction of the ideas and applications of elementary 'Set Theory' and 'Transformation Geometry'. It is expected that these will help in breaking down the barriers between the different branches of elementary mathematics *viz.* Arithmetic, Algebra, and Geometry which are usually treated as separate subjects in the traditional system. Further the young pupil will through them come in contact with some fundamental concepts which they will use considerably at a mature age when dealing with advanced mathematics. There is possibly some advantages in these approaches to the teaching of elementary mathematics, but the method is still under experiment and has not been fully adopted even in U.S.A. or U.K. Some other important aspects are also to be considered in this connection which are probably more valid in India than in foreign countries. We have to handle a large mass of school children and consequently also a large number of teachers who were not exposed to these new concepts. Moreover not even one per cent of the school children who will compulsorily take a course in Mathematics will choose to specialize in Higher Mathematics where they may have further opportunities to develop these 'new' concepts. In consequence the class room presentation will be ineffective in most cases and the large majority of young pupil will not derive any advantage from it in their later life. The objectives of general education and in particular those relating to Mathematics must depend on

the educational value of Mathematics as well as the practical values of Mathematics. There are certain phases of mathematics which are indispensable tools for an intelligent man and their acquisition by every child should be regarded as essential. Thus through arithmetic should come the concept of measurement and the calculations needed in the daily life of man. The familiarity with the more common geometric forms and their mensuration are essential needs of man. The use of graphs and the simpler notions of statistics are useful tools for the developments of the social life of the community. These subjects therefore must form an integral part of the general education at the school level. The 'new' concepts cannot be introduced by replacing some of these essential topics. The time available for instruction is limited and has to be shared reasonably with other subjects like languages, sciences including social science, and physical education. The rate of progress in the class room instruction should normally be based on the average standard of the class. The quantum of the subject that can be prescribed must be carefully estimated so that all topics may be built up in the proper sequence and nothing is left over from year to year. Moreover a spiral approach must be followed in the programme for mathematics instruction. The important topics are to be repeated in subsequent classes and with greater depth and abstraction. The mode of instruction should encourage the pupils to integrate mathematics learnt with life experiences, and lay more stress on understanding the subject rather than on memorising and naturally these will require more time for a proper progress. The syllabus should therefore prescribe only those topics which are the minimum that must be presented to the class in the proper sequence and at the same time express rigidly the domain in which the questions set for the examinations must lie. The teachers should always have full freedom in giving additional material whenever feasible depending mainly on the quality of his class. The willing and intelligent students will also then appreciate the trend of 'new' mathematics when they can learn them under no threat of the possibility of their knowledge being tested

in public examinations. On the contrary more emphasis should be laid on those aspects of the topics which can indicate their application in the day to day life of the community, and has always been neglected so far. This will help the large mass of students and make the class more lively.

Some historical developments of mathematics, particularly those relating to the contributions of the Hindu mathematicians will create considerable interest and inform the future generations about some interesting aspects of our proud heritage. Some informations about the life and works of internationally reputed scientists, whose works the students may come across in course of their studies, may also be interesting to the students and they will certainly get inspiration from those informations. These materials should be outside the scope of any public examination and the teachers may present them at suitable stages. Some of these informations have been given in Appendix III, which however, may be expanded whenever possible.

Our education system should provide for the intellectual needs of the students as future citizen. These needs however, may go on changing, still provision should be made to equip them with as many problems as may be appreciated by them. Personal problems in business and financial transactions in relation to taxes, insurance and annuities will be faced by almost all the adults, and the teachers may point out how the knowledge of elementary mathematics may be applied to solve them. Some indications of these types of 'consumer' mathematics have been given in Chapter III which the teachers may further expand and introduce in the class as problems and applications of the different principles included in the syllabus. Economic problems concerning supply and demand, production, transformation and demand, even by the use of some simple methods of 'Linear Programming' can be easily introduced in the school stage and some simple cases have been incorporated in Chapter III. The construction and interpretation of graphs when applied to life problems relating to public health, education, industry and such other problems connected with the life of the community will create interest in

the class and show the important applications of graphs and graphical representations. These have also been indicated in Chapter III for the convenience of the teachers.

It is certainly not possible to precisely lay down all the scopes of mathematics teaching at the school level, but only a few important ones have been indicated above. The teachers will certainly be able to extend them from time to time, from a study of their environments and the needs of the community.

CHAPTER II

DETAIL PROGRAMMING OF THE SYLLABUS AND TREATMENT OF THE SUBJECT MATTER

2.0. The proper organisation of a teaching programme for mathematics should help the students in expressing thoughts clearly and accurately and also help them in arriving at correct conclusions by careful interpretation of mathematical results through accurate and logical reasoning where necessary. In developing the different branches of mathematics more emphasis is laid on the logical and rigorous treatment of the subjects which are to be presented in the most effective way. It is also necessary to explain clearly the processes and principles which are the basic fundamentals of the subject. Before presenting a new topic its motivation should be explained, if possible, through concrete experiences of the student or through creation of a situation in which the basic concepts may be clear, and in that process attempts should be made to integrate mathematics with life experience. A spiral approach to the different topics is always useful, and a topic should not be finished once for all, but should be repeated again and again in greater depths and abstraction in later stages, thereby reinforcing the knowledge gained by the students in earlier stages. It is also desirable to encourage clear understanding of the subject and its different topics, instead of creating conditions which may help memorising them, even when understanding takes more time. The process of teaching should be from concrete to abstract and mathematics should appear to the students as an abstracting science and not an abstract science. To make the teaching efficient it is necessary to make repeated use of the fundamental concepts in new situations as this will help retention and mastery of the basic concepts. It is also equally important to show the applications of the basic concepts, wherever possible, as these will help in a proper understanding of

any rule or formula derived from them. These rules and formulae may then be used with confidence when it is time-saving, but the students will have no difficulty in realizing the various implications of the principles involved. Mathematics, like many other science subjects is a built-in subject and a proper sequence is to be maintained in its presentation. In framing the syllabus this aspect had to be satisfied. If for any reason the time schedule cannot be maintained at any stage, the relevant portions of the syllabus must on no account be skipped over, as this will ultimately make the teaching and its appreciation defective. Some books and teachers adopt the method of leaving more difficult material at the cost of rigour, others may prefer a slower rate of progress retaining also the difficult materials. It is however, desirable to present the materials which are correct and rigorous, even though it be necessary to defer their proofs till a later stage. In presenting any new material in the class it is necessary to explain clearly the fundamental processes involved and the important consequences and their relations if any, with the materials discussed earlier. This may be presented by the teacher without the active participation of the students which will no doubt be time-saving, but the success of this procedure depends on the ability of the class and the interest the students may get in the lectures. The other method may be to secure active participation of the students through questions and answers. Though this process may be more time-consuming it will have the advantage of getting more activity and co-operation from the class and will be more useful to the majority of the students. The time to be devoted to a topic however, will depend on the response of the class. The students should always be encouraged to apply his knowledge of the principles learned to the solution of a wide variety of problems, whose difficulty, length and variety must be properly chosen to suit the ability of the students. Usually the class contains a heterogeneous group and it is necessary to indicate a minimum assignment for the slower students and to suggest a few more difficult problems for the better students.

The utilitarian aspects of mathematics is one of the important

motivation for their study. Mathematics now is a necessary tool for a large group of students. A teacher of elementary mathematics should therefore stress on the applications of the subject to the different possible interests of the students. Segregation on the basis of individual needs is not possible and the syllabus must be based on a compromise between the possible needs and the ease with which they can be introduced. Only such applications which may be commonly understood and do not require much time in the explanation of the underlying principles and processes can be presented to the class. In Chapter III some possible applications have been shown which may create and maintain the interest of even the less able students in a class. For the realistic mind the practical use of any subject will vitalize the work and become one of the strongest motivating forces for seeking knowledge of the subject. Only a few of such uses have been indicated in Chapter III which may be found useful in later life by even those students who may not pursue any course in mathematics after leaving the school.

To cater to the needs of the brighter students the introduction of some 'modern' concepts will be no doubt useful. In the following sections some indications will be given as to what and where they can be conveniently introduced. The teacher will have to judge carefully, based on the quality of his class, whether introduction of these concepts may be useful. A good student becomes very annoyed, and rightly so, if the class instruction is inefficient. If an undue amount of time is spent on some topics it becomes a source of discontent and loss of interest, particularly to good students. One of the important functions of the teacher is to see that the class instruction is carried out efficiently and economically.

The main topics included in the syllabus for school mathematics are arithmetic, geometry and algebra. In the following sections are given brief outline lists of the principal topics which are to be presented to the students through text-books and class instructions. These are to be introduced in different classes in the proper sequence. Some of the topics are to be repeated in

successive classes when different parts are to be discussed and these have been marked with asterisks. The different sub-groups should be taken in proper sequence if necessary in successive classes. The subjects are to be presented in the traditional methods, but the equivalent modern concepts can be introduced whenever feasible, provided the class can receive them properly. The places where they can be introduced have been indicated in the following sections.

2.1. Concept of postulates and their uses :

In mathematics as in other fields, a starting point must be assumed on the basis of certain primitive concepts which cannot be defined in terms of simpler ideas. Such concepts remain undefined but their properties serve as co-ordinating factors in deducing other results. In each subject certain postulates are assumed which are fundamental and accepted without proof on the basis of experience. Their proof is either not possible or any proof given requires the assumption of some other property of an equivalent nature. A set of postulates, however, must be consistent and categorical. They are consistent if no postulate of the set contradicts the existence of any other. The set is categorical if each postulate is necessary and the set is complete so that this may not be deducible from any other and is sufficient in developing the subject. The different subjects in elementary mathematics arise from the methods employed in their developments and not in the object of study. Arithmetic is confined largely to the study of the fundamental operations with real positive numbers while algebra employs letters or symbols to represent numbers, although their values often remain unassigned. The number system then is extended to include negative real numbers, and complex numbers. The postulates in algebra are similar to those assumed in arithmetic. Certain common notions are however, employed throughout the development of all the branches of mathematics which are the following :—

(*a*) Two quantities equal to the same or equal quantities are equal.

(*b*) If equal quantities are added to or subtracted from equal quantities, the results are equal.

(*c*) If equal quantities are multiplied or divided by equal quantities the results are equal except that division by zero is not permissible.

Some more postulates are assumed which help in the development of the subject. The modern trend is to make a unified approach to all subjects but that may be difficult for presentation to the average school students. It is therefore desirable to continue the traditional methods in the development of the different subjects, at least in their early stages. In every case therefore the postulates should be carefully enunciated and the subject matter may then be developed by strict logical reasoning. In the following sections we give in brief outlines, the topics that should be covered in different branches of mathematics. We have also indicated the nature and position of new material which are not in the syllabus, that may be incorporated in order to enlighten at least the brighter students, with the trends in higher mathematics. Simultaneously the application of the principles demonstrated to problems which may appear in the daily life of the community should be stressed upon. For teaching purposes the order indicated may be altered if found convenient, but the entire material must be considered without any drop-outs.

2.2. Teaching material in Arithmetic :

The topics indicated in the syllabus are given below with some amplifications. Each of the sections may be further amplified suitably with a view to incorporate new ideas and historical informations whenever feasible. Some indications towards that have also been given in the subsequent sections. The details of the syllabus are as follows :

I. Revision of previous works : Basic operations with integers :

(*i*) The number system ; place value of digits.

(*ii*) The four simple rules : addition, subtraction, multiplication and division; simple applications.
(*iii*) Concept of prime numbers and factors.

II. Measurement of quantities :
(*i*) Units of length, areas and volumes.
(*ii*) Units of weight, time and money.

*III. H.C.F. and L.C.M. of numbers :
(*i*) By division.
(*ii*) By factorisation.
(*iii*) Relation between H.C.F. and L.C.M. of numbers.
(*iv*) H.C.F. and L.C.M. of vulgar fractions and decimal fractions.
(*v*) Problems.

IV. Operations with vulgar fractions and decimal fractions :
(*i*) Meaning and use of fractions and decimals.
(*ii*) Reduction of fractions.
(*iii*) Conversion of fractions to decimals and vice versa.
(*iv*) Addition, subtraction, multiplication and division of fractions and decimals.
(*v*) Problems.

*V. Extraction of square roots :
(*i*) Meaning of squares and square roots.
(*ii*) Extraction of square roots by factorisation.
(*iii*) Extraction of square roots by division.
(*iv*) Extraction of square roots of vulgar fractions and decimal fractions.
(*v*) Problems.

*VI. Unitary methods :
(*i*) Basic principles of unitary methods.
(*ii*) Application to problems on percentage.
(*iii*) Application to problems relating to profit and loss.
(*iv*) Application to problems on time and work, time and distance.
(*v*) Simple interest and Income Tax calculations.
(*vi*) Problems.

VII. Average :

(*i*) Meaning of average.
(*ii*) Bar graphs and histograms.
(*iii*) Problems.

VIII. Ratio and Proportions :

(*i*) Meaning and use of ratio and proportions.
(*ii*) Properties of a proportion.
(*iii*) Problems.

2.21. Supplementary reading materials in Arithmetic :

The topics mentioned in the previous section are well discussed in text-books following traditional lines. There are however, some materials which may be simultaneously presented to the students with a view to showing the rigour in the treatment of the subject and simultaneously giving an idea of the new developments in mathematics in recent years. Occasional references to the historical background, particularly of those where Indian mathematicians have contributions will add to the interest of the students. This will also give them some idea about the origin and evolution of mathematical principles and help in a proper understanding of the fundamental concepts. These have been indicated in a few cases and the teacher may decide as to how and when these informations can be presented before his class.

2.22. Postulates of arithmetic :

The following basic assumptions should be explained with examples after Section I (*ii*) :

(*i*) Unique existence law : For any two numbers their sum and product are uniquely determined (closure property). The concept of addition composition and multiplication composition may be introduced here.

It should be explained that the set of natural numbers is closed with respect to addition and multiplication but not closed with respect to subtraction and division.

(*ii*) Commutative law : Addition and multiplication are commutative.

(*iii*) Distributive law of multiplication over addition. A multiplying factor can be distributed with each of two or more terms in addition.

(*iv*) Associative law : Addition and multiplication are associative.

The concept of identity elements may be given and it may be shown that 0 is the identity element under addition for the set of rational numbers and 1 is the identity element for multiplication.

2.23. Decimal system and Binary numbers :

While discussing the number system, the concept of the positional numeral system with base 10 (or radix 10) should be explained. The idea may be generalized for any base b for which the basic symbols may be 0, 1, 2, ..., $(b-1)$. In that system any number N can be represented by the sequence of basic symbols

$$a_n a_{n-1} a_{n-2} \cdots\cdots a_2 a_1 a_0$$

where $0 \leqslant a_i < b$, $i=0, 1, 2 \ldots\ldots n$

and the magnitude of N is

$$a_n b^n + a_{n-1} b^{n-1} + a_{n-2} b^{n-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

Thus in the decimal system the number 4271 has the magnitude

$$4\times10^3+2\times10^2+7\times10^1+1.$$

As an extension to this, the concept of Binary numbers which are used in digital computers and switching circuits may be explained. In case of binary numbers the radix is 2 and the symbols are 0 and 1. Thus the binary number 1011 in binary system has in the decimal system the value

$$1\times2^3+0\times2^2+1\times2^1+1=11$$

and we can write this result in the form

$$(1011)_2=(11)_{10}$$

The Hindu-Arabic Numeral system may be discussed at this stage and the history of the invention of 0 may be introduced

at this stage. The relevant materials given in Appendix III will be interesting.

While discussing I (*iii*) the ideas of *set* or *class* may be presented. It may be explained how all prime factors of a given integer is a *subset* of the *set* containing all integer numbers equal to or less than the given integer. Thus all integers from 1 to 24 is a *set* containing 24 elements. 24 has prime factors 1, 2, 3 and they form a subset of the first set. This concept of sets and subsets will be useful during discussions of H.C.F. and L.C.M.

The concept of the operations of sets, their *union* and *intersection,* may be introduced here. The idea of representation by Venn diagram may also be introduced. The material presented in Appendix I may be useful in this connection.

2.24. Measurement of quantities in other units :

Although only metric measures are included in the syllabus, it may be useful to give the students some idea about the other measures which were in vogue in this country prior to the introduction of the metric system. The British units in the F.P.S. system or even the earlier units of Indian coins and weights should be discussed. In earlier literature these have been used and unless the students get some acquaintance with these units they may face difficulties in their daily life. Suitable conversion tables may be given even in text-books for purpose of references. Thus the relation between a foot and a metre or a pound, a seer and a kilogram should be explained.

2.25. H.C.F. and L.C.M. by using the knowledge of Sets :

The idea of operations on sets can be used in obtaining the H.C.F. and L.C.M. of specific pairs of natural numbers. If A, B denote the sets of factors of two numbers p and q respectively then

$$A \cap B$$

denotes the set of the common factors of p and q, which is a non-empty finite set, since A and B are also finite. It must

therefore have a greatest number and this greatest number is the unique H.C.F. of p and q.

Cor. : 1. Every common factor of two numbers is a factor of their H.C.F.

Ex. 1. Find the H.C.F. of p and q where $p=60$, $q=98$.
In this case

$A = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

$B = \{1, 2, 7, 14, 49, 98\}$

$\therefore$ $A \cap B=\{1, 2\}$

$\therefore$ The required H.C.F. is 2

By extending this method it is possible to obtain the H.C.F. of any set of natural numbers. Thus if p, q, r are three given numbers for which A, B, C are respectively the sets of their factors, then obviously

$$A \cap B \cap C$$

is a non-empty finite set. It must therefore have a greatest element which is the H.C.F. of the three given numbers.

Ex. 2. Find the H.C.F. of 24, 56, and 68

$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$B = \{1, 2, 4, 7, 8, 14, 28, 56\}$

$C = \{1, 2, 4, 17, 34, 68\}$

$\therefore$ $A \cap B \cap C=\{1, 2, 4\}$

$\therefore$ the H.C.F. required is 4.

The lowest of the common multiples of two numbers is called the lowest common multiple or the L.C.M. For any two numbers the L.C.M. must exist and is unique. We can also find them by using the results of the set theory. Thus if p, q be two numbers and A, B are the set of multiples of these numbers then

$A=\{p, 2p, 3p,........\}$ or $\{xp : x \in N\}$

$B=\{q, 2q, 3q,........\}$ or $\{xq : x \in N\}$

where N is the set of natural numbers 1, 2, 3,.................

Then $\qquad A \cap B$

is a non-empty set which must have a least number or element. This lowest number is the L.C.M. of p and q.

Ex. 3. Find the L.C.M. of 6 and 9.

$$A = \{6, 12, 18, 24, 36, ...\}$$
$$B = \{9, 18, 27, 36, ...\}$$
$$\therefore \qquad A \cap B = \{18, 36, ...\}$$

Hence the L.C.M. of 6 and 9 is 18.

Extending this idea it may be seen that the L.C.M. of any finite set of numbers, is the lowest member of the intersection set of the sets of multiples of the given numbers.

Ex. 4. Find the L.C.M. of 16, 24, 30.

The application of set theory is useful only when the numbers are small and few. In other cases the traditional methods of obtaining H.C.F. by repeated division and of L.C.M. and H.C.F. by expressing the given numbers as product of primes are more useful. The finding of H.C.F. and L.C.M. by using the results of set theory is however, illustrative of the ideas of set theory.

2.26. Fractions and decimals :

After discussing the different properties of fractions we may bring the concept of *order-denseness* of the set of fractions and the notion of *betweenness.* Thus if

$$\frac{p}{q}, \frac{r}{s}$$

are two fractions such that

$$\frac{p}{q} < \frac{r}{s};\ p, q, r, s \in N$$

i.e., p, q, r, s are natural numbers, then a fraction u/v always exists which lies between p/q and r/s, i.e.

$$\frac{p}{q} < \frac{u}{v} < \frac{r}{s}.$$

It is easy to show that we may take

$$\frac{u}{v}=\frac{1}{2}\left(\frac{p}{q}+\frac{r}{s}\right).$$

Similarly between u/v and r/s there must exist a fraction. We thus have the following result :

Between two different fractions there lies an infinite number of fractions. This property of fractions is called *Order-dense.* This property does not hold in case of natural numbers.

Cor. : The set of natural numbers is a sub-set of a set of fractions.

From this concept it may be possible to go to the idea of rational numbers and their representation.

In sections V—VIII more emphasis may be laid to problems which will be useful in the life of the community and occur in the daily life of the people. In many cases algebraic methods may be useful and some formulae may be established for use in numerical problems. Wherever possible algebraic methods should be used in solving problems in arithmetic.

One of the important features of arithmetic is the wide variety of its applications. The utilitarian value of arithmetic should always impress the student and serve as a motivating force. It is therefore necessary to choose more practical problems for teaching the applications of the principles discussed in the different sections.

2.3. Teaching material in Algebra :

The topics presented in the syllabus are given below with some amplifications. Some duplication of the material with that included in arithmetic is inevitable, but their treatment will be on a more general level. Here also suggestions have been made as to the possibility of introducing the concepts of 'new' mathematics at suitable places. It is for the teacher to decide as to whether these should be presented in the class and if so to what extent. Indications have also been given as to the possible application of the principles to 'consumer mathematics' which will create

considerable interest in the minds of the young pupils. The details of the syllabus are as follows :

I. (*i*) The use of symbols to generalise arithmetical problems.
(*ii*) The number system—positive and negative numbers.
(*iii*) Fundamental operations with signed numbers.
(*iv*) Postulates of algebra : The laws.

*II. Applications of the fundamental operations :
(*i*) The four fundamental operations with polynomials.
(*ii*) Formulae and their applications.

*III. Factors :
(*i*) Simple factors.
(*ii*) Quadratic factors.
(*iii*) H.C.F. and L.C.M. by factorization.

*IV. Solving problems using equations and inequations.
(*i*) Simple linear equations and inequations.
(*ii*) Simultaneous linear equations and inequations.
(*iii*) Quadratic equations and inequations.

V. Graphs :
(*i*) Rectangular co-ordinate system—plotting of points.
(*ii*) Graphing linear equations.
(*iii*) Graphical solution of equations.

2.31. Supplementary reading material in Algebra :

In Section I it will be interesting to generalise the idea of arithmetic and thus explain the need of algebra. Similarly in developing the principles of algebra a frequent reference to their possible applications to arithmetical problems, which the students have already learnt, will be useful. The concepts of negative numbers, constants, and variables represented by algebraic quantities, are to be introduced at this stage. The rules for the four fundamental operations with both positive and negative numbers are to be explained. At this stage it will be necessary to demonstrate, by suitable examples, why division by zero is not permissible.

The students will then appreciate the proper operations with signed numbers or variables and realize the difference between arithmetic and algebra and the larger scope of algebra.

The postulates of algebra may now be explained which are similar to those discussed before in Sec. 2.22. They may be expressed in terms of algebraic numbers and variables.

It is possible at this stage to introduce the idea of a *set* While in classical algebra we consider algebra of real numbers, the more general concept would be the algebra of a *class* which brings in the concept of a *set*. The 'algebra of sets' is fundamental in understanding 'modern mathematics'. The preliminary definitions of a *set, sub-set,* their *union* and *intersection* may be explained with suitable examples as indicated in Appendix I. The concept and use of Venn diagram may be introduced here. The fundamental laws or postulates of the *set* theory may be explained without difficulty at this stage. The scope of the few extra laws valid in the algebra of sets over those valid in the algebra of real numbers should be explained with some indications as to why they are not taken in the algebra of real numbers. The scope of the algebra of sets and their possible applications to problems of arithmetic, geometry and probability theory may be interesting to the students. A few examples have been incorporated in Appendix I.

In Section II the definition of a polynomial and its degree should be explained. The four operations with polynomials, the use and removal of brackets and their effect on the signs of the individual terms of a polynomial should be explained. After deriving the various formulae it will be interesting if their equivalent representations in the case of polynomials of second degree are presented.

In Section IV we may start by explaining that the motivation for solving equations lies in solving problems. It will therefore be useful to construct suitable problems connecting one, two, or three variables which can be expressed in terms of equations that may be linear or quadratic or may contain both. It may then be interesting to devise various methods for the solution of these

equations. The idea of 'equations' and 'inequations' should be introduced at this stage.

In this section the idea of *set* and the set of rational numbers may be repeated with a view to explaining the 'solution set' of equations. The concepts of 'statements', 'open statements' and 'solution set' may be introduced here. The nature of the 'Truth table' can also be explained in this connection. The meaning and use of 'ordered pairs', and 'order relation' may be explained here with suitable examples. The 'truth set' of an 'equation' or of an 'inequation' should be explained with reference to simple examples. The idea of the 'domain of a variable' in an 'open statement' may be explained at this stage. The relation between an equation and an identity may be explained with reference to their corresponding 'truth sets'. The solution set of a linear equation containing two variables may be obtained and the concept of an 'ordered pair' may be explained in that connection. The 'truth set' for some linear equations may be obtained for purpose of illustration.

Ex. 1. Find the truth set of the equation

$$3x-4y+5=0.$$

Since

$$3x+5=4y,$$

then

$$\frac{3x+5}{4}=y.$$

By giving to x different values we can get the values of y. Thus when

$$x = 1, \quad y = 2$$
$$x = 2, \quad y = 11/4$$
$$x = 3, \quad y = 7/2$$
$$x = 4, \quad y = 17/4$$
$$x = 5, \quad y = 5$$

Thus the following *ordered pairs* form the truth set of the given equation :

$$(1, 2), (2, 11/4), (3, 7/2), (4, 17/4), (5, 5)$$

If there be two simultaneous linear equations in two variables, then the truth set of each equation may be obtained. It can be seen that the truth set of the two equations may contain either one element, or infinite number of elements or may be void. Accordingly the pair of equations have (i) a unique solution, (ii) infinite number of solution, or (iii) no solution. In other words, the pair of equations are (i) consistent, (ii) dependent, or (iii) inconsistent.

These ideas may be explained with reference to some more examples.

The traditional methods of solving simultaneous equations by elimination of all variables except one and the graphical method of solution may be discussed at this stage.

The problems of arithmetic relating to (i) time and distance, (ii) profit and loss, (iii) time and work, (iv) stocks and shares which are useful applications may be transcribed in terms of suitable equations and their solution can be obtained. Some other methods of solving equations and inequations and of obtaining their truth sets are given below.

2.32. Use of determinants in solving linear equations :

It may be useful at this stage to introduce some elementary ideas relating to *determinants* and their use in solving simultaneous equations.

Let us consider two equations

$$a_1x+b_1y+c_1=0 \qquad .... \quad (1)$$

$$a_2x+b_2y+c_2=0 \qquad .... \quad (2)$$

By eliminating y [multiplying equation (1) by b_2 and equation (2) by b_1 and subtracting] we get

$$(a_1b_2-a_2b_1)x+(b_2c_1-b_1c_2)=0.$$

Thus

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}.$$

Thus x exists if $(a_1b_2-a_2b_1)\neq 0$. This is sometimes expressed in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

and the expression on the left is called a determinant. It consists of two rows and two columns, and in its expanded form each term is the product of two quantities. It is therefore said to be a determinant of second order. In a similar manner a determinant of third order and its equivalent expression may be written as follows :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2) + c_1(a_2b_3-a_3b_2)$$

The following properties of the determinants may be demonstrated with simple examples.

(*i*) The value of the determinant is not altered by changing the rows into columns, and columns into rows.

(*ii*) If two adjacent rows or columns of a determinant are interchanged, the sign of the determinant is changed but its magnitude remains unaltered.

(*iii*) If two rows or two columns of a determinant are identical the determinant vanishes.

(*iv*) If each element in any row, or in any column, is multiplied by the same factor, then the resulting determinant is equal to the original determinant multiplied by the same factor.

We can in general define in like manner a determinant D of nth order which contains n rows and n columns. If a_r be an

element then the determinant obtained by removing the row and column in which a_r occurs is also a determinant of $(n-1)$th order. If that determinant is represented by A_r then A_r is called the minor of a_r. The value of the determinant D is then given by the relation,

$$D=a_1A_1-a_2A_2+a_3A_3-a_4A_4+\cdots\cdots+(-1)^{n-1}a_nA_n,$$

where a_1, a_2, ..., a_n, are the elements of the first row or column taken in the serial order. We can take the elements of any other row or column by using (*ii*) above.

By using the above properties of the determinants it may be shown that the solution of the equations

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$

is given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Similarly the solution of the equations

$$a_1x+b_1y+c_1z+d_1=0$$
$$a_2x+b_2y+c_2z+d_2=0$$
$$a_3x+b_3y+c_3z+d_3=0$$

is given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

It may be interesting to the students if it be explained to them that the process is a general one and can be extended to the solution of n linear equations containing n independent variables.

In the case of the pair of simultaneous equations containing

two variables x, y, discussed above, it may now be explained that if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

the equations are consistent, and have a unique solution.

If $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \neq 0,$$

the equations are inconsistent and the truth set is void, and if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$$

the equations are dependent and the truth set is infinite.

2.33. Quadratic inequations :

In solving quadratic inequations it may be useful to bring in the idea of truth sets. Thus in finding the truth sets of the inequality

$$ax^2+bx+c > 0$$

or $$ax^2+bx+c < 0,$$

we use the rule that the product of two numbers is positive if and only if the numbers are both positive or both negative.

Thus if

$$ax^2+bx+c = a(x-\alpha)(x-\beta)$$

then $$ax^2+bx+c > 0$$

if (taking $a > 0$)

$$\{x : (x-\alpha) > 0 \text{ and } (x-\beta) > 0\}$$

or $$\{x : (x-\alpha) < 0 \text{ and } (x-\beta) < 0\}$$

Now if $\alpha > \beta$, then the above conditions give

$$\{x : x > \alpha\} \cup \{x : x < \beta\}$$

Similarly

$$ax^2+bx+c < 0$$

if (taking $a > 0$)

$$\{x : (x-\alpha) < 0 \text{ and } (x-\beta) > 0\}$$

or $$\{x : (x-\alpha) > 0 \text{ and } (x-\beta) < 0\}$$

But $x > \alpha$ and $x < \beta$ is false as $\alpha > \beta$.
Hence the required solution is

$$\{x : \beta < x < \alpha\}$$

2.34. Graphs of a linear equation:

In Section V while discussing *graphs* it may be shown that the truth set of a linear equation containing two variables lie on a straight line in the plane defined by the two variables. The concept of co-ordinates referred to a rectangular system of axes may be explained at this stage. It should be demonstrated that the co-ordinates of any point on a straight line joining two given points satisfy a linear equation. Hence conversely the graph of a linear equation is always a straight line. It may also be explained by introducing the measure of the area of a triangle in terms of a determinant of third order whose elements are the co-ordinates of the vertices of the triangle.

2.341. Graphs of a pair of linear equations:

The graphs of two linear equations

$$a_1x+b_1y+c_1 = 0$$
$$a_2x+b_2y+c_2 = 0$$

with suitable numerical values for a_1, b_1, c_1 and a_2, b_2, c_2 may be drawn. It may be shown that the two lines will

(*i*) intersect at a point if $a_1b_2-a_2b_1 \neq 0$,

(*ii*) be parallel if $a_1b_2-a_2b_1=0$ and $a_1c_2-a_2c_1 \neq 0$,

(*iii*) coincide if $a_1b_2-a_2b_1=0$ and $a_1c_2-a_2c_1=0$,

corresponding to the three cases considered in section 2.32.

It may be interesting to show that a quadratic equation which may have two linear factors represent a pair of straight lines.

2.342. Graphs of linear inequation :

Consider the inequation

$$ax+by+c > 0. \qquad \dots \quad (1)$$

We have seen that the equation

$$ax+by+c=0, \qquad \dots \quad (2)$$

represents a straight line whose slope and position will depend on the magnitude and signs of a, b, and c. If therefore (x_0, y_0) be a point on the line (2) then

$$ax_0+by_0+c=0 \quad \text{or} \quad by_0=-ax_0-c.$$

Hence if y be the ordinate of a point satisfying (1) and having the abscissa x_0 then

$$by > -ax_0-c$$

and hence

$$b(y-y_0) > 0$$

Thus if b is positive then $y > y_0$ and if b is negative then $y < y_0$.

Hence if $b>0$, the inequation (1) will represent all points of the plane above the line represented by (2) and if $b<0$, then it will represent all points below the line represented by (2). The concept above and below is related to the position of the y-axis with respect to the x-axis and in the above case it has been assumed that if y is positive then the corresponding point will be above the x-axis.

If however

$$ax+by+c \geqslant 0$$

then the points also include those lying on the line

$$ax+by+c=0.$$

Ex. 1. Draw the graphs of the following inequations

(*i*) $2x-4y+3 \geqslant 0$,

(*ii*) $3x-2y+2 \leqslant 0$.

It may be noted that the inequation

$$ax+by+c < 0$$

is equivalent to the inequation

$$-ax-by-c > 0$$

and thus all possible cases of inequations are covered by the above discussions.

2.343. Graphs of a pair of inequations:

If there are a pair of inequations:

$$U \equiv a_1x+b_1y+c_1 > 0,$$
$$V \equiv a_2x+b_2y+c_2 > 0,$$

it is possible to obtain the set of points represented by the combinations of U and V. Thus the combination U *or* V *or* both is given by

$$U \cup V,$$

and similarly the combination of U *and* V is given by

$$U \cap V.$$

Thus we may first draw the graphs of

$$U = 0 \text{ and } V = 0,$$

and then obtain the set of points representing $U > 0$ and $V > 0$ and finally the union or intersection of these sets to get the solution of the pair of inequations. The same procedure may be extended to the combination of even more than two inequations.

2.35. Bar graphs and Histogram:

The nature of statistical data and their representation may be introduced here. The reasons why they are represented by bar graphs and not by continuous curves as in the case of a continuous function, may be explained. The physical significance of

histogram may also be explained. These have been described briefly in Chapter III.

2.36. Applications to practical problems:

In recent times considerable applications are made of linear equations, their solution sets and graphs in problems related to industry, business and defence, in addition to their use in problems related to natural sciences. Some of the problems of industry and business management are designated as 'Operations Research' and some of these problems can be easily solved by using the methods of 'Linear Programming'. A simple description of this method has been given in Chapter III and may usefully be presented to the students at this stage. A large number of students may find it useful in their later life and will certainly find the topic interesting.

2.37. Quadratic equations and inequations:

The solution of a quadratic equation

$$ax^2+bx+c=0,$$

may be expressed in the form

$$x=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a},$$

and the different cases as to the nature of the roots depending on whether b^2 is greater than, equal to or less than $4ac$, should be discussed. It should also be shown that a quadratic equation can have two and only two roots.

A pair of equations, of which one is quadratic may be solved by first eliminating one of the variables.

In finding the truth sets of a quadratic inequation we may first put the quadratic function as a product of two linear factors as shown in an earlier section.

2.38. Contributions of ancient Hindu Mathematicians:

The most significant contribution of the ancient Indian Mathematicians, which probably has contributed the most towards

the general progress of mathematics in particular and science in general, is the invention of the *principle of position* in writing numbers. Generally the numerals are called Arabic numerals, but it should be called the "Hindu" numerals as the Arabs borrowed them from the Hindus. The nine figures for writing the units are supposed to have been introduced earlier but the sign of zero and the principle of position is of later origin. The date and the name of the inventor is not known precisely. Probably the zero was inrtoduced at about the time of Aryabhatta who was born in 476 A.D.

Aryabhatta made some interesting contributions in solving linear as well as quadratic equations. Linear equations with two variables were also discussed by Bhaskara, Brahmagupta, Narayana, Gangadhara and Sridhara. Some indications of their works along with a brief description about their life is given in Appendix III and may be presented to the students while discussing the topics in which they made important contributions.

2.4. Teaching material in Geometry:

Geometry is by nature sequential and it is essential that the students are made to understand the fundamental concepts which form the core of the subject. Starting with the undefined elements, the concept of points, lines, planes, and space, the whole subject of synthetic geometry is built upon the foundation based on the postulates and definitions which are introduced as and when necessity arises. The synthetic geometry of Euclid develops those properties of geometric figures which remain invariant under motion. Starting with some definitions and postulates we should proceed to establish the facts of geometry through logical proofs. Postulates are however, not made up at random. .They describe fundamental properties of space which are obvious. In a similar way the undefined terms indicated above are suggested by physical objects.

Students should be encouraged to the frequent use of geometrical instruments *viz.* compass, protractor and ruler. A neat and precise drawing of the figure often helps in solving a geometrical problem.

The proof of any proposition in geometry starts with the hypothesis and by precise logical deduction the conclusions are arrived at. In consequence the practice in solving geometrical problems helps the students in developing logical arguments. On the other hand the effect of changing one or more postulates on the results obtained in a theorem will indicate the real significance of the different postulates. A considerable amount of practice in solving a large number of problems following a theorem should be encouraged.

The topics indicated in the syllabus are given below with some necessary amplifications.

I. Properties of geometric figures :
- (*i*) Concept of points, lines and planes through common objects and their properties.
- (*ii*) Line segments and angles.
- (*iii*) Idea of solid bodies and plane figures through common objects and models, construction of paper models.
- (*iv*) Relation between volumes, faces and edges of a rectangular parallelopiped, a cube and a tetrahedron.

*II. Basic ideas of transformation geometry :
- (*i*) Simple idea of reflection and its consequences.
- (*ii*) Studies on the effects of reflection through paper folding.
- (*iii*) Idea of symmetry in geometrical figures.
- (*iv*) Concept of translation and rotation and their uses in establishing geometrical congruences and similarities.

*III. Geometrical drawing :
- (*i*) Geometrical instruments and their uses.
- (*ii*) Measurement of line segments and angles.
- (*iii*) Geometrical constructions involving straight lines, angles, triangles, quadrilaterals and parallelograms.

*IV. Theorems and their applications :
- (*i*) The congruence theorems on triangles and angles.
- (*ii*) Properties of parallel and perpendicular lines.
- (*iii*) Theorems on circles and parallelograms.

(*iv*) Inequalities of sides and angles of triangles.
(*v*) Theorems on concurrent lines.
(*vi*) Similar triangles.
(*vii*) Pythagoras' theorem.

V. Properties of space figures :
(*i*) Perimeter and area of a rectangle, a triangle and a circle.
(*ii*) Surface and volume of a rectangular parallelopiped, a cylinder, and a sphere.
(*iii*) Practical applications.

VI. Elementary concepts on Trigonometry :
(*i*) Trigonometrical ratios using properties of triangles.
(*ii*) Easy applications.

2.41. Supplementary reading material in geometry :

In recent years some changes have taken place in the manner of presentation of plane geometry and attempts are being made to examine how far even elementary geometry can be presented from an advanced standpoint. Some amount of informal geometry is now introduced at the early stages through which the students can easily acquaint themselves with some of the basic facts and simple constructions which appear in an advanced course of geometry. It is therefore necessary to present the definitions of the basic terms and the postulates in a more general way so that they may be used also without modifications even at the later stages.

In Section I, while illustrating the undefined elements, it may be possible to explain the space, planes and lines as suitable 'set' of points. This will then give precise definitions of *segments*, *rays* and even triangles as a 'set' suitably defined. We can define the angle as an *ordered pair* and bring in the concept of *directed angles*. The idea of *betweenness* may be introduced without difficulty at this stage. This will facilitate in presenting the ideas on congruences for segments and angles. If attempts are made to visualize the mathematical concepts through models and drawings they will give a better impression on the pupils. Thus paper models of bodies may be shown as much as is possible and this

will also help in explaining the formulae for the measure of the volumes of simple solids and that of the area of their faces.

The congruence property of angles and triangles may conveniently be demonstrated by paper folding or paper cutting. Theoretically those are problems of 'transformation geometry' and it may be useful if these concepts, as indicated in Appendix II, are introduced while discussing the topics mentioned under Section II. By using the simple and powerful modern ideas of 'transformation geometry' we can formulate the congruence concept of arbitrary figures. We can also develop the ideas of *rigid motion* which will be useful in the proper realisation of advance courses on geometry. This method in a better way replaces the crude superposition arguments offered by Euclid.

The congruence of angles can also be shown in a similar way and it may be explained that it is an *equivalence relation.* The concept of the magnitude and sign of an angle should be explained with illustrations. Similarly the congruence of triangles should mean that one triangle is an exact copy of the other. The vertices are to be properly paired. Thus if triangles ABC and A′B′C′ are congruent then we have,

$$A \leftrightarrow A', \ B \leftrightarrow B', \ C \leftrightarrow C',$$

and

$$\angle A \leftrightarrow \angle A', \angle B \leftrightarrow \angle B', \angle C \leftrightarrow \angle C',$$

so that there is one-one correspondence between the vertices of $\triangle$ ABC and $\triangle$ A′B′C′.

As an application of the congruence relation we can establish the angle-bisector theorem or the isosceles triangle theorem which states as follows :

If two sides of a triangle are congruent, then the angles opposite to these sides are congruent.

Given a triangle ABC, if in it AB=AC, then

$$\angle B \cong \angle C$$

Proof : Consider the correspondence

$$ABC \leftrightarrow ACB$$

between $\triangle$ ABC and itself. Under this correspondence, we have

$$\overline{AB} \leftrightarrow \overline{AC},\ \overline{AC} \leftrightarrow \overline{AB},\ \angle A \leftrightarrow \angle A.$$

Thus the two sides and the included angle of $\triangle ABC$ are congruent to the parts that correspond them. Hence by the S.A.S. postulate, we get

$$\triangle ABC \cong \triangle ACB$$

that is, the correspondence

$$ABC \leftrightarrow ACB$$

is a congruence. But by the definition of a congruence all pairs of corresponding parts of the triangles are congruent. Hence we have

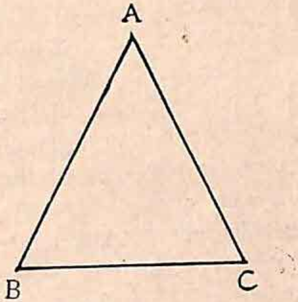

Fig. 2.1. An isosceles triangle.

$$\angle B \cong \angle C$$

as these angles are corresponding parts.

Although in the case of a triangle a congruence is established by a one-one correspondence between its vertices which leads to the SSS theorem, the same is not sufficient for polygons. Thus a square and a rhombus can have sides of the same length still they may not be congruent. The definition of congruency requires that if $F \leftrightarrow F'$ be a one-one correspondence between two sets of points F, F′, such that

$$P \leftrightarrow P'\ ;\ Q \leftrightarrow Q'$$

always, and for every P and Q on F and P′ and Q′ in F′ then

$$PQ = P'Q'$$

and F and F′ are congruent, and we may write

$$F \cong F'$$

Thus the four vertices of a quadrilateral ABCD do not yield four but six segments *viz.* AB, AC, AD, BC, BD, CD. If they are equal to the corresponding segments of the other quadrilateral then they are congruent. In general it is essential that the pairs of points of two infinite sets should preserve distance. It may

be shown that this is sufficient to preserve the angle measure between any two segments intersecting at a point.

Thus we observe that a motion or transformation between two point sets F and F′ is a rigid motion if it preserves distances and hence is a congruence between F and F′. Thus two figures are congruent if they can be made to coincide by a rigid motion.

The basic types of rigid motions are translations in which no point is fixed, rotations in which the axis of rotation is fixed and reflections in a plane in which each point of the plane is fixed. Some simple cases of these have been given in Appendix II and may be explained to the students. This may give the students an idea about some modern methods of treating the problems of geometry which is concerned with the study of transformations (rigid and non-rigid) of n-dimensional spaces. A simple case of non-rigid motion is enlargements and has also been explained in Appendix II and may be interesting to the students.

In Section III some idea of approximation may be given. Even when a geometrical figure is constructed according to the stipulated conditions the figures may not be exact and there is a possibility of the existence of difference between the works of different students. The dependence of that on the imprecise nature of the instruments used and also on the personal error of the students may be explained.

In Section IV the theorems are to be presented in their proper sequence, so that they can always be derived on the basis of results already established.

A number of problems should be discussed following each theorem, as that will help in a proper realisation of the principles established in the theorem and at the same time help in developing logical arguments. Some occasional reference to the earlier works of Indian Mathematicians may be of interest.

Chapter V deals mainly with the problems of mensuration. Suitable examples may always be given for practice, which may show their utilitarian aspects. The field surveyor uses these results in measuring field areas and the manner in which they do it has been shown in Chapter III.

CHAPTER III

USE OF MATHEMATICS IN COMMUNITY LIFE

3.0. It is desirable to put some emphasis on the utilitarian aspects of elementary mathematics in the day to day life of the individual and the community. Applications of mathematics to problems of the home and community should be duly emphasised during class instructions. This will help in a better way the proper realization of the progress as well as utility of the various projects now in operation in the country. In planning for buildings and roads accurate maps are required which involve mathematics, particularly scale drawing and surveying. Local business problems may also be used to create interest and motivation. Careful decisions in buying goods and commodities help in the progress of daily life, and mathematics properly presented may be effective in these matters. A large portion of the business in progress is done on credit and the businessmen should carefully compare the cost of instalment buying and that of large-scale purchase after securing loans at high rates of interest. The mathematics of trades, business and professions should be presented at suitable stages and the students will get more interest in it when they find their applications, as a majority of them will eventually seek their livelihood within these fields of work. The problem of investment is one of mathematics. The comparison of various investments involve the principles of simple and compound *interest.* The information and calculations underlying the *insurance* business which has now taken a major role in the life of the community, will create interest and love for mathematics amongst students who may be exposed to these informations. The problem of taxation has become an important factor in the life of every man and it will be useful if the students are given reasonable preparation to cope with such problems which they may have to face later in their life. 'Linear programming' and 'Operations research' are now

used with advantage in problems of management of business and industry. When the instructions relating to solution of equations and graphs have been completed the students may be shown how their knowledge may be usefully applied also in planning for business and industry.

Limitations of time available and the average standard of the class may restrict introduction of some of the topics indicated above and the teacher will have to decide carefully in the matter. Works of the type indicated above will help in preparing the students to accept opportunities for their own improvement and are intimately associated with the social, economic and industrial problems of the country. A few of the problems indicated above are discussed below in some details.

3.1. Percentage :

The basic principles of percentage and some of their common uses are included in the curriculum. It will be interesting if the students are shown how *percentage* is used in several types of business transactions. They may be acquainted with the following terms commonly used in business circles.

A business man may *borrow* money from different sources or from a bank and pay *interest* for using the money for a definite length of time, and the interest is calculated at a certain *per cent* of the money borrowed. Stores and other establishments sometimes offer *discounts* at different rates in order to attract more buyers and increase sales. Salesmen are appointed often on *commission* basis and they are given certain percentage of the amount they get from the sale proceeds. Merchants who import articles from abroad, have to pay *customs duties* at a certain *per cent.* Property owners pay *taxes* which are also calculated in *per cent.* The rules used for the calculations of *Interest, Discounts, Commissions, Taxes,* and *Duties* are similar, except for the addition of one or two simple calculations, and may be illustrated with some examples similar to those indicated below.

3.11. Simple and commpound interest :

The problems reltaing to *simple interest* are discussed in text-books. The idea of *compound interest* which is always used in bank transactions may be introduced here in the following form :

Let P be the principal and r the percentage rate of interest per year. If m be the number of times in a year the interest is to be compounded, then the amount A, accumulated at the end of n years, will be given by

$$A=P\left(1+\frac{r}{100m}\right)^{nm},$$

and the amount of interest thus compounded in n years will be

$$A-P=P\left[\left(1+\frac{r}{100m}\right)^{nm}-1\right].$$

Note : Only small values of m and n should be taken so that the calculations may be done without the use of logarithms, as the students are not expected to know that at this stage.

The following problems may illustrate the methods of calculations.

Ex. 1. A person borrowed from the bank a sum of Rs. 25,000.00 for 3 years at 9% compound interest, compounded half yearly. Calculate the amount of interest he has to pay at the end of 3 years.

Ex. 2. A person borrowed from a bank a certain amount and invested that in his industry. After 2 years he repaid his debt and paid interest at the rate of 8% compounded every month. Calculate what percentage of the amount he invested was paid as interest.

3.2. Discount and Brokerage :

The methods of calculating the amounts of discount and brokerage may be explained with the following examples.

Ex. 3. Rice was purchased at a total cost of Rs. 2,500.00 and the purchaser was allowed a discount of $4\frac{1}{2}$% for cash pay-

ment. What would be the cost price if the discount was not allowed ?

Ex. 4. A buyer may purchase an article with Rs. 7,500.00 when a single discount of 20% is allowed. How much will he be required to pay if two successive discounts of 15% and 5% are allowed ?

Ex. 5. A broker sold a building for a certain sum of money and remitted Rs. 39,200.00 to the owner after deducting his commission at 3%. Calculate the commission received by the broker.

Ex. 6. Calculate the value of the stock that can be bought for Rs. 4,841.00 allowing the broker a commission of 3%.

Ex. 7. A merchant imported 400 tins of powder milk each weighing one kg. and invoiced at Rs. 7.50 a tin. The duty was 30% *ad valorem* and in addition Rs. 10.50 a kg. What was the total duty paid ?

Ex. 8. A factory worker working 48 hours a week received 50 Paise per hour. His week's wages were raised to Rs. 28.50. Calculate the per cent raise in the wages of the worker.

3.3. Income-tax calculations :

In the assessment of Income-Tax of individuals the rates of taxes in different slabs of income are to be explained. The different rebates normally allowed are also to be explained. It then reduces to a problem of percentage calculation and may be easily carried out. The following examples may illustrate the methods of calculations.

Ex. 1. For the assessment year 1972-73 the prescribed rate of income-tax was as follows :

On total annual income

(*i*) not exceeding Rs. 5,000	*Nil*
(*ii*) for the next Rs. 5,000	10% of the income
(*iii*) for the next Rs. 5,000	17% " " "
(*iv*) for the next Rs. 5,000	23% " " "
(*v*) for the next Rs. 5,000	30% " " "

In addition a surcharge of 10% of the amount of income-tax was also payable. The tax-payer, however, was eligible to deduct the following from the gross annual income for calculating the tax :

(*i*) cost of books purchased for self use up to Rs. 500.

(*ii*) Rs. 2,400.00 if the assessee owns a car used for the purpose of employment during the whole year.

(*iii*) A rebate for the amount of contribution to Provident Fund and/or the premium paid on the policy of insurance on own life, limited up to 30% of the gross total income and calculated at the following rate :

The whole of the first Rs. 1,000 of the qualifying amount plus 50% of the next Rs. 4,000 plus 40% of the remainder of the qualifying amount.

(*iv*) Income from Bank interest up to Rs. 3,000.00.

(a) Calculate the amount of income-tax payable by A whose income and other relevant informations are given below :

(*i*) Salary and allowances	— Rs. 2,000.00 per month
(*ii*) Contribution to P. F. and Life Insurance Premium	— Rs. 4,000.00 per year
(*iii*) Cost of books purchased during the year	— Rs. 360.00
(*iv*) Maintains a car which is used for the purpose of employment	

(b) Calculate the amount of income-tax payable by B whose income and other relevant informations are given below :

(*i*) Salary and allowances	— Rs. 500.00 per month
(*ii*) Contribution to P. F. and Life Insurance premium during the year	— Rs. 2,500.00
(*iii*) Income from Bank interest	— Rs. 550.00

3.4. Annuities and earnings from Bank deposits :

The calculations of the returns available from different types of investments are important particularly when provision is to be

made for the old age. The banks allow a higher rate of interest for fixed deposits for a specified number of years. It is also possible to purchase 'bonds' from the 'Unit Trust of India' which allows annual dividends. A simple calculation will indicate which method of investment may be preferable, judged from the amount of available return. These may be illustrated by the following examples.

Ex. 1. A person wanted to invest a sum of Rs. 5,200.00. The sale price of the bonds of Rs. 10.00 *nominal value* issued by the Unit Trust of India was Rs. 10.40 on July 1, when the money was to be invested. The Unit Trust gave a dividend of 8.50% for the previous year. The banks offered to give an interest at the rate of 7% per year for a fixed deposit for two years, in which the interest was payable every month, which could be deposited in the savings bank fetching interest at the rate of 4% per annum. Calculate the annual incomes on the two investments assuming that the rate of interest and dividend continued unchanged, and indicate which of the investments would be profitable to the investor.

Ex. 2. A person executed a Life Insurance Policy at the age of 35 years. It was an 'endowment' policy for 15 years 'without profit' for a sum of Rs. 10,000.00. He had to pay an annual premium of Rs. 700.00. Examine how much would he gain, if instead of taking the Insurance Policy he would deposit the full premium every year in the P.O. Savings Bank, which would allow 4% interest, when he receives the money at the end of 15 years. If the policy taken was 'with profit' and the L.I.C. declared a bonus at the rate of Rs. 15.00 per thousand per year, what amount the insurer would get on maturity of the policy ? (Assume that the insurer survives the period of 15 years.)

3.5. Linear Programming :

The mathematical model of Linear Programming and its various applications to industry and business management have been developed in recent years. The fundamental concepts can be illustrated in two-dimensional space based on the knowledge of algebra

and geometry learnt in school mathematics. It will, therefore, be of interest to the school students to see how they can apply their knowledge of mathematics in making some decisions in problems relating to industry and business, which are being developed in their neighbourhood.

Linear Programming, however, deals with the general problem in which a set of equations or constraints in the following form are given

$$a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n \leqslant c_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n \leqslant c_2$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$
$$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\cdots+a_{nn}x_n \leqslant c_n$$

and it is required to obtain the values of $x_1, x_2 \ldots x_n$, which may be non-negative, so that a linear form

$$b_1x_1+b_2x_2+b_3x_3+\cdots\cdots+b_nx_n=\text{B}$$

may be a maximum. It may also be possible to relax the condition that the n variables must be non-negative. In a similar way we can also obtain the values of $x_1, x_2, \ldots, x_n$, so that B may be a minimum. Both the cases are covered if we find the values of the variables by optimizing B. The general problem is beyond the scope of school students, but simple cases may be discussed which will give some preliminary idea about the principles of Linear Programming and also at the same time show their applications to some problems of industry and business management. This is illustrated by the following example.

Ex. 1. A steel cabinet maker undertakes the manufacture of two different articles, (1) big almirahs with glass doors and (2) small filing cabinets for office use. Both the articles are to pass through three sections. (I) Machine section putting the components in proper sizes, (II) Fitting section, where the components are properly joined and (III) Polishing and painting section where the manufacture is completed. The man-hours required in each

department for each component along with the total available man-hours per week in each department are given in the following table together with the amount of profit to be derived per unit.

	Departments I	Departments II	Departments III	Profit per unit
Almirahs : hours per unit	12	6	3	Rs. 100
Filing cabinet „ „ „	5	4	4	Rs. 40
Total man-hours available	120	66	48	

The manager has to decide as to the number of almirahs and filing cabinets to be manufactured per week depending on the following factors or objectives :

(*a*) To gain maximum profit.

(*b*) To manufacture maximum units in order to cater to a large number of customers, and

(*c*) To arrange for maximum utilization of the employees of the firm.

Calculate the profit expected in each case and obtain the feasible decision in the management of the business.

Let A be the number of almirahs and F the number of filing cabinets to be manufactured each week. The equations of constraints are therefore

$$\begin{aligned} 12A+5F &\leqslant 120 \\ 6A+4F &\leqslant 66 \\ 3A+4F &\leqslant 48 \end{aligned} \qquad \text{....} \quad (1)$$

Also A and F must be non-negative.

The total profit per week is

$$100A+40F \qquad \text{....} \quad (2)$$

The total units produced per week is

$$A+F \qquad \text{....} \quad (3)$$

The time spent for the almirahs is

$$12A+6A+3A = 21A$$

and the time spent for the filing cabinet is

$$5F+4F+4F = 13F$$

Thus the total utilization of facilities (man-hours) is

$$21A+13F \quad .... \quad (4)$$

Thus the problem reduces to the determination of A and F subject to the constraints (1) and maximization (or optimization) of either (2) or (3) or (4) depending on the objectives (*a*), (*b*) or (*c*).

In figure 3.1 we have drawn the graphs of the three equations derived from (1) having their points of intersections at $(8\frac{1}{3}, 4)$, $(6, 7\frac{1}{2})$ and $(7\frac{3}{11}, 6\frac{6}{11})$.

In figure 3.1 we have also drawn the first objective function defined by (2) *viz.*

$$100A+40F = 20C$$

$$5A+2F = C$$

for some value of C.

For different values of C this will give a set of parallel lines and as C increases, it moves away from the origin. The set of solutions giving A and F are enclosed by the three lines (I, II, III) drawn in Fig. 3.1, and the two axes of A and F. In a similar way the objective function defined by (3) is

$$A+F = C$$

and that defined by (4) is

$$21A+13F = C$$

These have also been drawn in Fig. 3.1 for some suitable values of C. Thus maximising the first objective function the solution comes to (taking A and F as integers)

$$A = 9, \qquad F = 2,$$

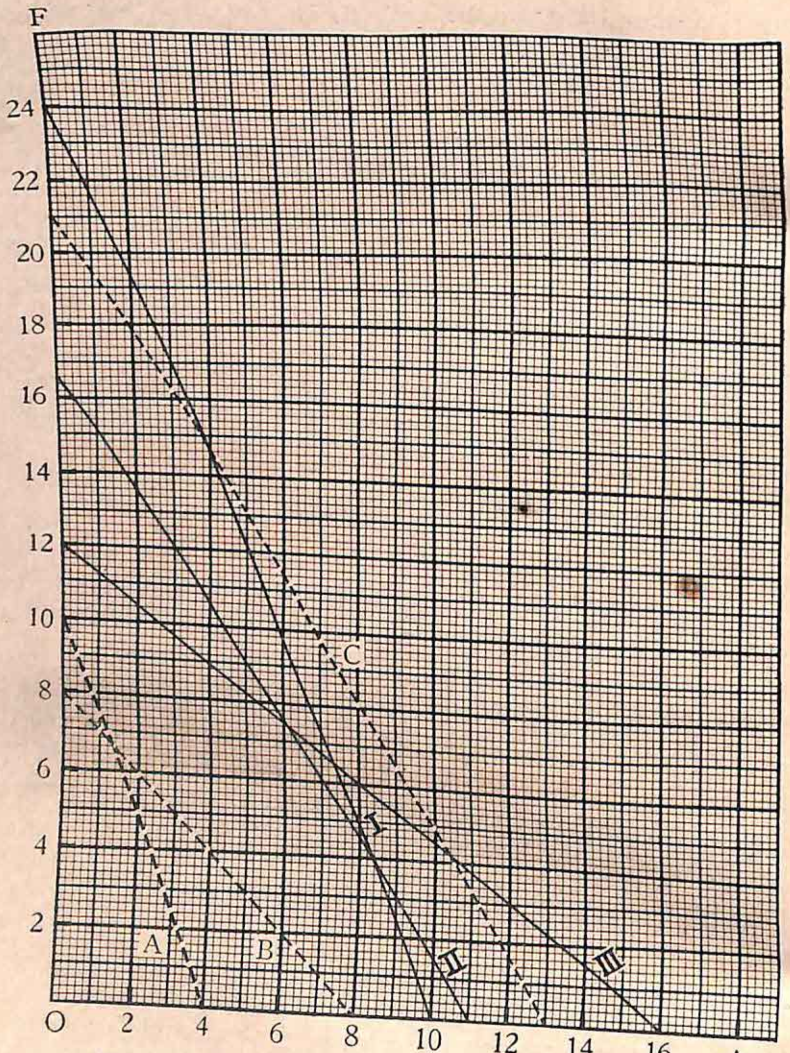

Fig. 3.1. Graphs representing constraints and objectives.

and hence the profit per week is Rs. 980.00. Similarly under the second objective the solution becomes

$$A = 6, \qquad F = 7,$$

and hence the profit per week is Rs. 880.00. Similarly under the third objective the solution becomes

$$A = 8, \qquad F = 4,$$

and hence the profit per week is Rs. 960.00, but the total man-hours utilized per week is 220.

It is now for the manufacturing concern to make a policy decision as to whether to make maximum profit, or to produce large quantity of articles for the benefit of the country or to antagonise the employees and invite labour trouble by using maximum man-hours for the work.

Ex. 2. Assume that only four nutrients for cattle have been isolated, namely, A, B, C, and D. The Department of Agriculture specifies that the minimum daily requirements per animal are :

2 lbs. of A, 4 lbs. of B, 4 lbs. of C and 3 lbs. of D

Only two feeds P and Q are available. The analyses of P and Q are as follows :

	P (per cent)	Q (per cent)
A	15	50
B	30	20
C	20	20
D	35	10

P costs Rs. 2.00 per kg. and Q Re. 1.00 per kg. Derive the least cost feeding program.

Ex. 3. Three different baby foods A, B, C are each made of

three components a, b, c. The amounts required to make 10 quintals of each are given below :

	A	B	C	Total available per week (quintals)
a	3	2	4	240
b	3	5	2	300
c	4	3	4	350

(*i*) How much of each product should be made if all the raw materials are to be used up ? What would the total profit be if A yields Rs. 50 a quintal while B and C yields Rs. 30 a quintal.

(*ii*) Calculate the profit made if 350 quintals of A and 180 quintals of B and C are produced. Can you find a better production scheme which may yield more profit ?

3.6. Graphical representation :

In algebra we have seen how variables are linked or related in some way. We have also observed how line graphs are drawn to show diagrammatically the relation of one variable with the other. The simple diagrams of continuous curves represent continuous functions. The advantage of graphs is that one can see certain conditions at a glance, while it may take a considerable time to get the same information by actual calculation. In some cases the variables concerned may be discrete and the line graphs are not convenient representations. It is possible to represent them by Bar-graphs or in Histograms. Several statistical data are represented in this form with convenience.

Each point in a line graph depends on two known quantities, one to be taken as abscissa and the other as the ordinate. The line obtained by connecting the various points thus plotted gives the line graph. Whether the curve joining the consecutive points

should be a straight line or a smooth curve depends on the nature of the function and may not be known at all in several cases. Thus the bi-hourly temperature of a patient in a hospital when recorded and plotted in a chart are joined by straight lines, though that may not give a correct indication of the temperature at the intermediate stages. But the bi-hourly records of the atmospheric pressure or temperature may be joined by a smooth continuous curve which may be more near to reality. For purpose of graphical representation we have only to define a set of values assumed by the independent variable x, say, and for each x there should correspond one or more values of the dependent variable y, then y is defined as a function of x in its domain. It is however, not always necessary to specify the mathematical relation between x and y. A mathematical equation connecting x and y may not even exist. A function may be considered as being equivalent to a table in which values of x and y are given. Many data in statistics come under this general definition of function. When the annual income of an industry is given in a table for a number of years, the income may be plotted against the year. If these points are joined by straight lines they will give a visual idea about the relative incomes in different years, but certainly do not represent the function at intermediate points. On the time axis a year should really be represented by an interval and not by a point, but the earnings actually spread over the whole year are here represented as a lump sum and concentrated at one point of time. The points on the broken line for intermediate points have no significance. A similar situation holds with discrete variates. These are better represented by bar-graphs or histograms.

3.61. Bar-graphs :

In drawing Bar-graphs also we take a horizontal axis and a vertical axis and mark them according to the data furnished in the problem. We draw bars of a length that will correspond to the amounts involved. No set rules are prescribed for choosing units and they depend on the domain of the variates. A few

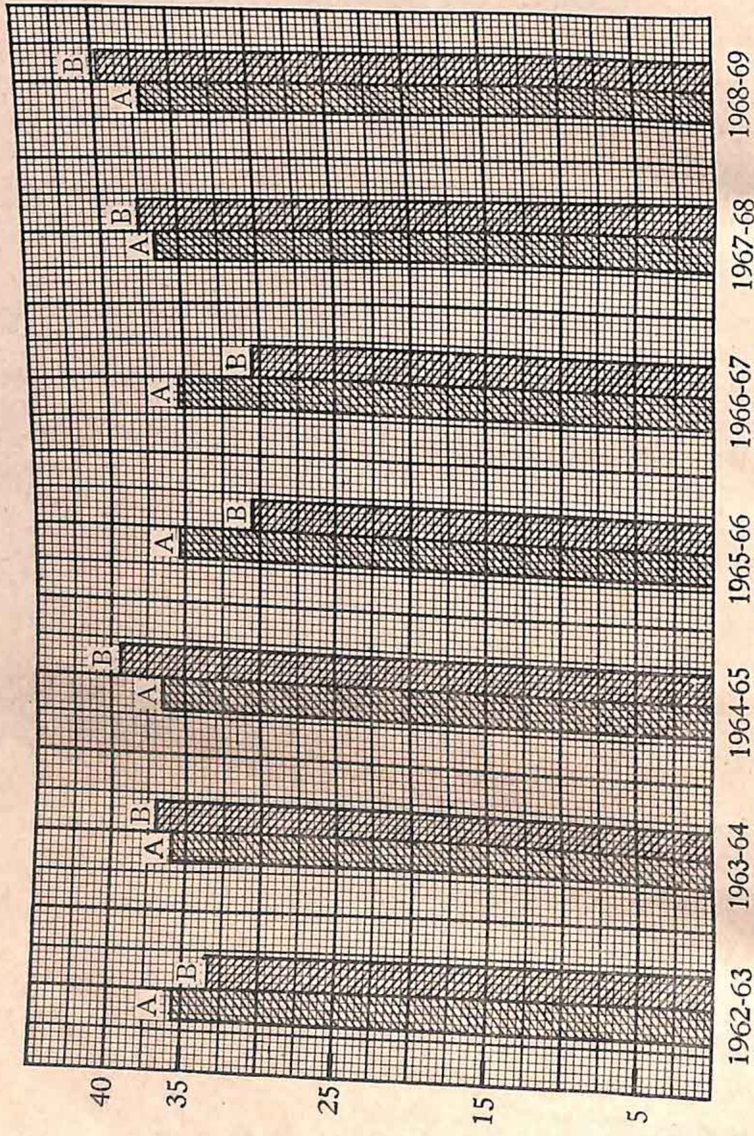

Fig. 3.2. Quantity of rice produced in India (B), and the area under rice cultivation (A) in different years. A in million Hectare, and B in ten million quintals.

examples are given below to show how different types of data can be represented graphically.

Ex. 1. Show by a Bar-graph the following data* giving the quantity of rice produced in India along with the area under rice cultivation during the seven years ending on March 31, 1969.

Year		Area under rice cultivation in 1,000 Hectare	Yield of rice in 10,000 quintals
1962-63	....	35,695	33,217
1963-64	....	35,809	36,998
1964-65	....	36,462	39,308
1965-66	....	35,273	30,655
1966-67	....	35,251	30,438
1967-68	....	36,437	37,612
1968-69	....	36,967	39,761

Ex. 2. Show by a Bar-graph the following data* giving the quantity of wheat produced in India, along with the area under wheat cultivation during the seven years ending on March 31, 1969.

Year		Area under wheat cultivation in 1,000 Hectare	Yield of wheat in 10,000 quintals
1962-63	....	12,927	10,776
1963-64	....	13,590	9,853
1964-65	....	13,499	12,257
1965-66	....	13,422	10,424
1966-67	....	12,656	11,393
1967-68	....	12,838	16,540
1968-69	....	14,998	18,651

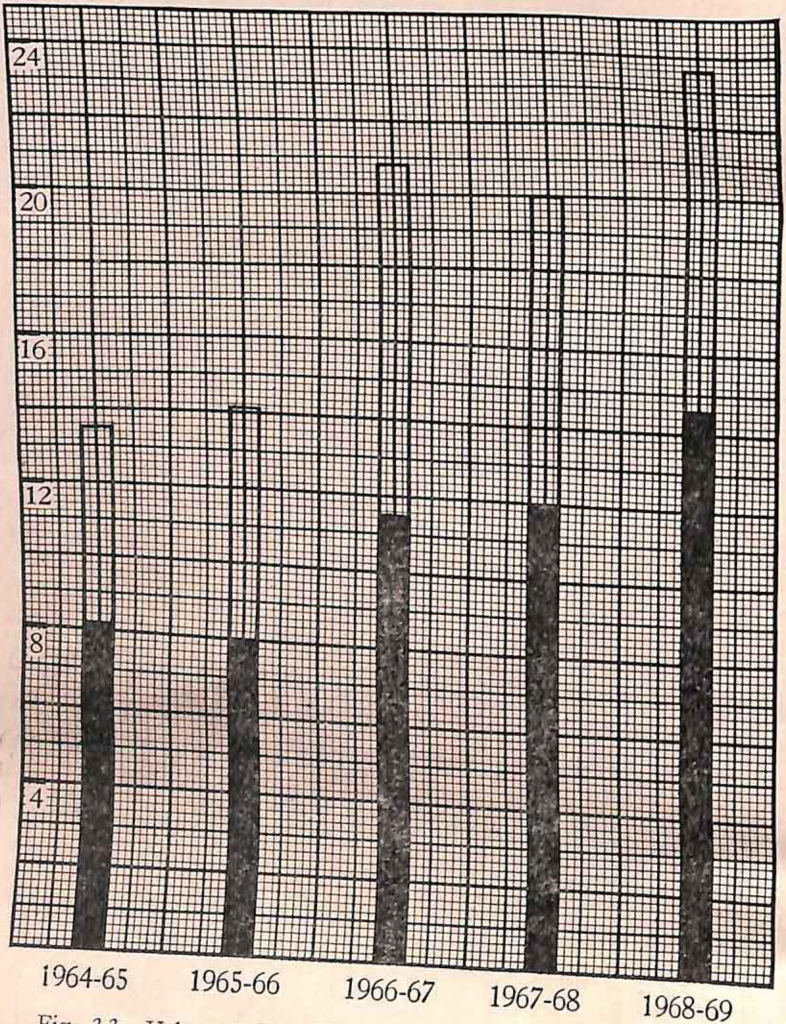

Fig. 3.3. Value of Import, Export (in black), and Balance of Trade (in white) of India for the five consecutive years ending on 31st March, 1969, in hundreds of crores of Rupees.

Also represent graphically the proportionate yield of wheat and rice per Hectare during the seven years indicated in Ex. 1 and Ex. 2.

Ex. 3. By means of a single bar-graph for each year during the five years ending on 31st March, 1969, show the value of imports, exports and balance of trade of India as given by the following table :

Value of total trade in merchandise by
sea, air and land*
(In Lakhs of Rupees)

Year	1964-65	1965-66	1966-67	1967-68	1968-69
Import	1349,03	1408,53	2078,36	2007,61	2360,02
Export	816,30	805,64	1156,54	1198,69	1458,87

In this case each year is represented by the same length in the horizontal axis. The imports have been represented by the entire length of the bar and exports by a shaded portion. The balance of trade will then be represented by the unshaded portion of the entire bar, so that three items may be indicated on a single bar.

Ex. 4. The table below shows the distribution of marks obtained by the students of a class in Mathematics. Represent it by a bar-graph :

Marks range		No. of students
0 — 10	....	2
11 — 20	....	5
21 — 30	....	20
31 — 40	....	30
41 — 50	....	35
51 — 60	....	31
61 — 70	....	19
71 — 80	....	5
81 — 90	....	2
91 — 100	....	1

* Data taken from : "Statistical Abstract India 1969", published by Central Statistical Organisation, Department of Statistics, Cabinet Secretariate, Government of India.

Ex. 5. The following table* shows the number of deaths in cholera and small pox in hospitals in West Bengal and Tamil Nadu in different years. Compare the mortality rates in the two states and also that between the two causes by means of a suitable graphical representation.

Year	West Bengal deaths			Tamil Nadu deaths	
	Cholera	Small pox		Cholera	Small pox
1958	1,878	283	....	355	188
1959	468	63	....	278	1,271
1960	589	21	....	63	260
1961	504	20	....	6	16
1962	1,112	78	....	10	71
1963	1,136	461	....	56	341
1964	433	94	....	642	160
1965	292	94	....	674	36

The total population according to 1961 census in West Bengal was 18,599,144 and in Tamil Nadu 16,327,135.

Ex. 6. The expenditure on recognized educational institutions in some states of India incurred by different bodies in 1965-66 are as follows* (in thousand rupees) :

State	From Government funds	Fees	Local bodies	Endowments
Maharashtra	335,640	116,013	52,283	45,394
Tamil Nadu	214,735	34,866	91,677	24,974
Uttar Pradesh	281,894	120,367	37,873	42,665
West Bengal	250,075	130,425	8,237	25,134

Represent the data by a suitable bar-graph.

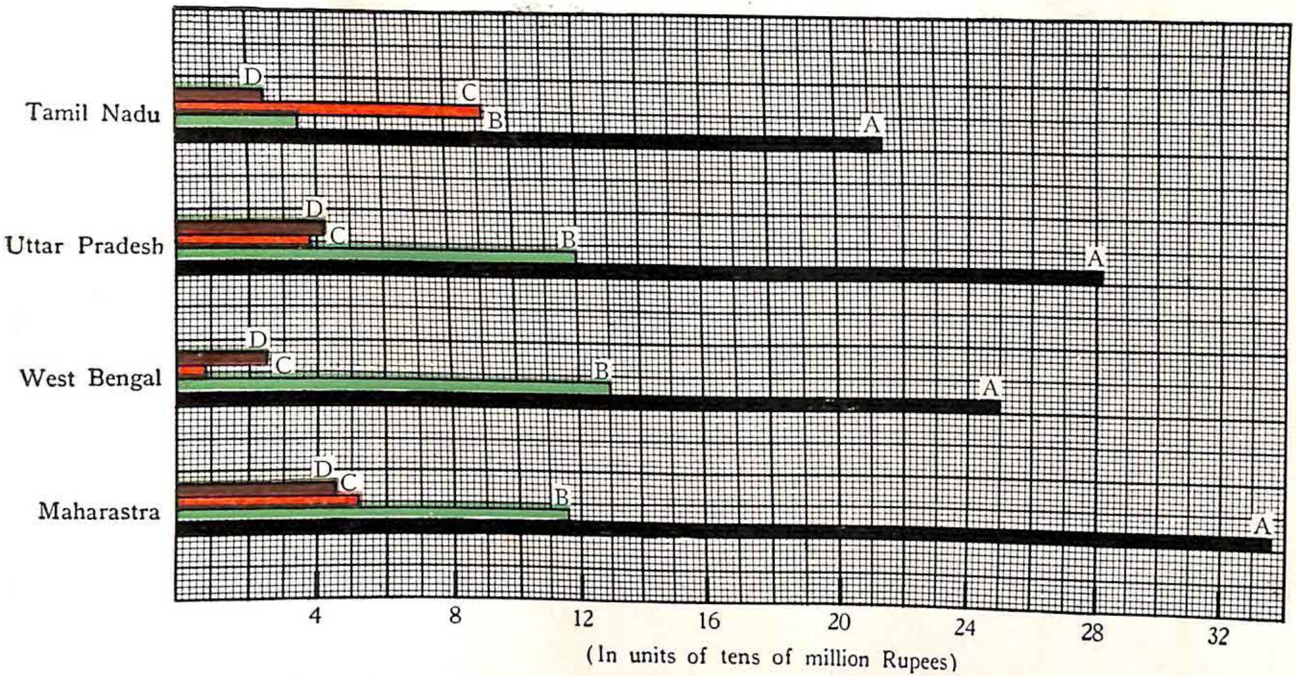

Fig. 3.4. Comparison of expenditures on educational institutions in four different states of India, incurred by different bodies : A—From Government funds, B—From fees, C—From Local bodies, D—From endowments.

3.62. Histograms :

A histogram is a set of rectangles with bases along the intervals between class boundaries placed on the horizontal axis, whose areas are proportional to the number or frequency in the corresponding classes. If the class intervals are equal the heights of the rectangles are also proportional to the numbers or frequencies in the corresponding classes. This method of representation of statistical data is very common and useful in the study of the distribution of values of a discrete variate. In a histogram the rectangles are all adjacent, since the bases cover the intervals between class boundaries and not class limits. In a bar-graph, on the other hand the spacing and width of the bars are arbitrary, and it is only the height that counts.

Ex. 1. Obtain the histogram corresponding to the data given in Ex. 4 of art. 3.61.

In this case if *Cm* represents the class marks and *Cb* the class boundaries then we have :

Class limits (Marks range)	Class boundaries (*Cb*)	Class marks (*Cm*)	
0 — 10	10.5	5.5	2
11 — 20	20.5	15.5	5
21 — 30	30.5	25.5	20
31 — 40	40.5	35.5	30
41 — 50	50.5	45.5	35
51 — 60	60.5	55.5	31
61 — 70	70.5	65.5	19
71 — 80	80.5	75.5	5
81 — 90	90.5	85.5	2
91 — 100	100	95.5	1

The intervals as determined by the class boundaries are adjacent, but as no marks can lie on a boundary there can be no ambiguity about the class to which any mark may belong.

The corresponding histogram is shown in Fig. 3.5. A curve has been superimposed on the histogram which is the fre-

quency curve showing the statistical distribution of the variates under consideration and the total area under this curve, like the total area enclosed by the histogram, represents the total frequency.

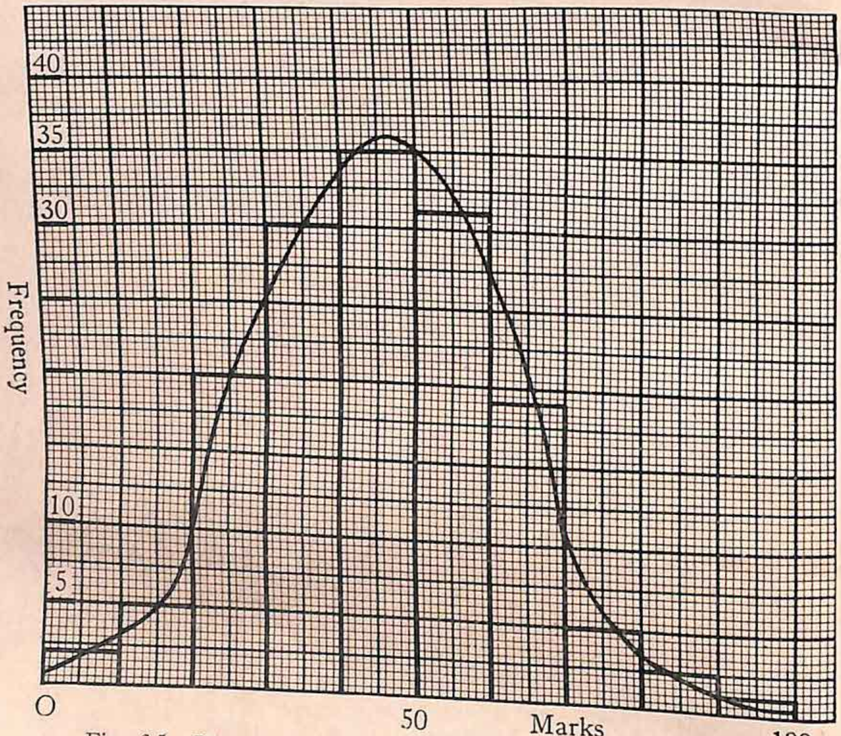

Fig. 3.5. Histogram showing distribution of marks obtained by the students in a class. The frequency curve showing the statistical distribution of the variates is also shown.

A reference to the histogram helps in defining the terms *Mean*, *Median*, and *Mode*, very often used in statistics. Thus if a distribution is represented by a histogram an ordinate through the median divides the area into two equal parts. Similarly an

ordinate through the mean passes through the centroid of the area. Similarly an ordinate through the mode (if there is only one mode) passes through the highest point of the frequency curve which fits the distribution. It, therefore, follows that if the frequency distribution is symmetrical then all three of these measures of location would coincide.

3.7. The surveyor's field book :

In geometrical treatments of different plane figures we have seen how to calculate the area of triangles, rectangles, circles etc. It is possible to use this knowledge in finding the areas of irregular rectilinear figures and explain the methods used by

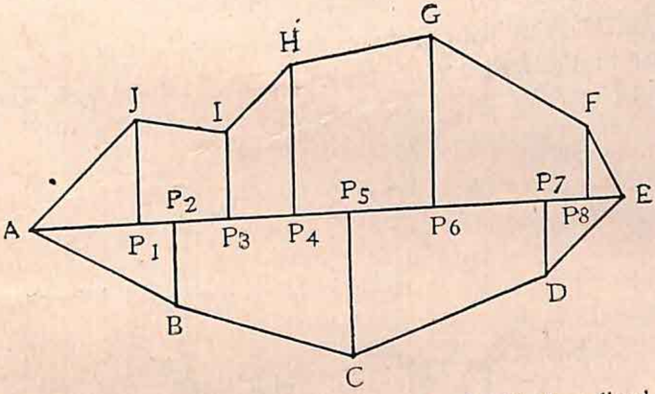

Fig. 3.6. The boundary of a plot of land with 'base lines' and offsets as taken by surveyors.

surveyors in recording and estimating the area of a plot of ground.

Consider an irregular area in the form of a polygon as shown in Fig. 3.6.

Join AE and from the other vertices drop perpendiculars on AE. The figure is thus divided into a number of traingles and

trapeziums, whose areas can be found out in terms of the lengths of their sides. The sum of these areas then give the measure of the plot. Here AE is defined as the *base line* and the perpendiculars JP_1, BP_2 etc. are called the *offsets*. The area of a plot of the ground can be expressed in terms of parts of *base line* and *offsets*. Distances along and perpendicular to the base line are measured by the surveyors by a chain and recorded in a *field book*.

Each page of the *Field Book* is divided into three columns. The central column contains the measurements made along the base line, the side columns record the measurements made along the offsets. The ends of the base line are called *stations*. The surveyor usually starts his entry from the bottom of the central column and writes upwards.

The following pattern shows the record in a field book corresponding to the measure of a plot of land whose sketch is also given in Fig. 3.7.

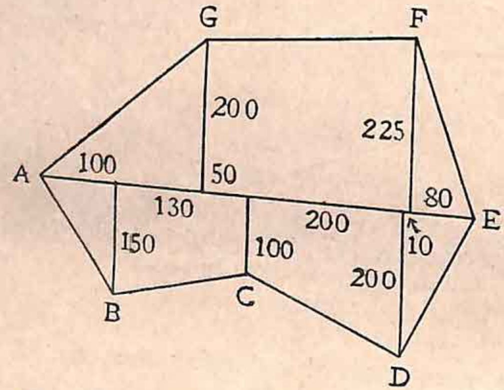

Fig. 3.7. The measures of offsets and their positions in a plot of land corresponding to entries made in a surveyor's Field book.

	Metres	
	To ⊙ E	
	80	
to ⊙ F 225	10	
	200	200 to ⊙ D
	50	100 to ⊙ C
to ⊙ G 200	130	
	100	150 to ⊙ B
	From ⊙ A	go to North

Thus the base line starts at A and runs North and thereby measures the offsets on the right and left as the case may be. The area of the plot then is :

$$\tfrac{1}{2}[100\times150+(100+130)\times200+(150+100)180$$
$$+(200+225)(50+210)+(100+200)200$$
$$+(80+10)\times200+80\times225]$$
$$=\tfrac{1}{2}[15000+46000+45000+110500+60000+18000+18000]$$
$$=1{,}56{,}250 \text{ Sq. metres.}$$

Ex. 1. A surveyor's Field-book has the following records. Draw a plan of the field and find its area.

	Metres	
	To ⊙ Q	
	140	
to ⊙ D 40	60	
	75	to ⊙ B 120
to ⊙ E 65	110	
	40	to ⊙ A 70
	From ⊙ P	go to East

Ex. 2. A paddy field has the sketch given below, in which all lengths shown in the diagram are in metres.

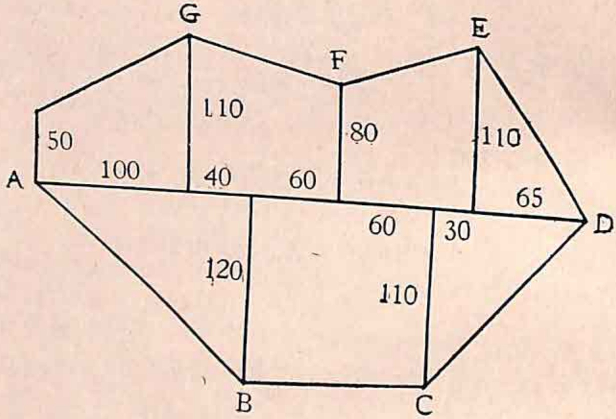

Fig. 3.8. The sketch of a paddy field with measures of offsets and positions of the offsets.

Give the entries in the surveyor's Field-book and also calculate the area of the field.

APPENDIX I

THEORY OF SETS AND BOOLEAN ALGEBRA

A. 1.0. Logic is an important tool in the development of Mathematics but the starting point in such developments has always been a few axioms whose validity is accepted without question and even without examining whether they conform to reality. By using these axioms and applying deductive logic, different theorems are established which remain valid only so long as the axioms hold and may be incorrect under different axioms. A significant example on this point can be seen in the axioms and some theorems of (i) the Euclidean geometry, (ii) the non-Euclidean geometry and (iii) the Riemannian geometry. The parallel postulate of the Euclidean geometry was changed by Lobachevsky and he obtained the non-Euclidean geometry. Riemann again changed both of them and got a new geometry and some of the results obtained by them are contradictory. It is thus evident that all mathematical truths are only relative and remain so as long as the basic axioms remain valid.

In the thirties of the present century a new and abstract approach has been made which has led to new results and has also developed new areas of mathematics. It has further helped in unifying Algebra, Analysis, Geometry and other branches of mathematics which in the traditional approach were considered as distinct. This new mathematics is based on some fundamental concepts, namely, *Sets, Relations, Mappings,* and *Operations,* and the different branches of 'classical' mathematics depend on their characteristic sets and the assumed structural relations between the elements of the set. In algebra we study the set of real numbers, while in geometry we take the set of points, lines and planes and base further discussions on postulates of parallelism and perpendicularity. The new algebraic system can thus be described as a set of objects along with some operations for

combining them. In the following sections we consider *sets* in general and the basic concepts of the algebra of *sets*. Such an axiomatic treatment of algebra will illustrate the methods used in the development of any mathematical system and also help in properly understanding the algebra of numbers which in the traditional methods, is often treated as a mass of unrelated rules and examples and not as one logical system.

Along with the different characters of the sets which appear in mathematics we have also to prescribe the rules of combinations, which are the structural relations between the elements of the set. Thus in the algebra of the real numbers we have the different sets and there are four rules of combinations, namely, addition, subtraction, multiplication, and division. In the algebra of sets we shall observe that there are only two rules of combinations, namely, *Union*, and *Intersection*. In the following sections we propose to discuss only some elementary principles of the algebra of sets which may with advantage be introduced in suitable places, in course of teaching elementary mathematics in schools.

A. 1.1. Notations and definitions in the Algebra of Sets :

The idea of a set is inherent in the concepts of familiar classes *viz.* (i) the set of integers with prescribed conditions, (ii) the set of all obtuse-angled triangles, (iii) the set of lines perpendicular to a given plane, (iv) the set of all books in a library, (v) the set of all pupils in the class room, and so on. In mathematics there are certain terms for which a rigid definition is not possible. Thus in geometry a precise definition of point, line and plane is not possible but we have some intuitive notion about these terms and they can be correctly used with confidence. Similarly in the algebra of sets the words *element* and *set* are also taken as undefined. Intuitively the elements of a set are taken as the basic objects which form the sets. Thus in the set of all positive integers, the integers 2, 5, 14 etc. are some elements. Similarly in the set of all pupils in a class, each individual student is an element of the set. There is a standard set of notations used in the algebra of sets, mainly because they enable us to be

both brief and exact. Usually capital letters are used to denote sets and small letters are used to denote the elements. Thus A, B, C, X, ... may be used to denote sets, while a, b, c, x, y, z, etc. may be used to denote elements of a set or sets.

If a be an element of a set A then we may write,

$$a \in A$$

and we may read 'a belongs to A'. If a does not belong to A we may write,

$$a \notin A.$$

An element of a set is not repeated.

If every element of a set A is also an element of another set B, then A is called a *subset* of B, and we may write,

$$A \subset B \quad \text{or} \quad B \supset A.$$

If B contains some elements which are not elements of A, then A is a proper subset of B. If however,

$$A = B,$$

then also the above condition holds and we may say that every set is a subset of itself and we may write

$$A \subseteq B.$$

This notation allows for the possibility that $A = B$. It is also evident that if $A = B$, then all elements of A are also the elements of B, and we may have both

$$A \subset B \quad \text{and} \quad B \subset A.$$

A set having no elements is defined as a *null set*. It is therefore a subset of every set. We may also indicate that a set A is a null set by saying that it is *empty*. A set consisting of all elements under discussion is defined as the *Universal Set*. Thus every set is a subset of the Universal set. It is sometimes denoted by U or 1.

A set is said to be finite if it is empty or consists of n elements, where n is a positive integer; otherwise it is said to be infinite. A

set may be specified by listing all the elements and we may do so by using the symbol { }. Thus if x, y, z, are the elements of a set S, we may take

$$S=\{x, y, z\}.$$

We can also use the following notation when the elements follow a prescribed rule. Thus if S be a set and A be a subset of S we can take,

$$A=\{a \in S \mid P(a)\},$$

where all the elements a of the set A satisfy the condition $P(a)$. Thus if S be the set of integers and A the subset of positive integers, then

$$A=\{a \in S \mid a > 0\}$$

The complement of a set is defined as another set consisting of all elements of the Universal set which are not elements of the original set. The complement of a set A is denoted by A′ and thus

$$A'=\{x : x \notin A\} \quad \text{or} \quad \{x \in U \mid x \notin A\}.$$

A. 1.11. Product Set and Ordered pair :

If A and B are two sets, then the product set $A \times B$ (or A cross B) consists of all pairs (a, b) where

$$a \in A \quad \text{and} \quad b \in B.$$

Thus if

$$A=\{1, 2, 3\} \quad \text{and} \quad B=\{5, 7\}$$

then

$$A \times B=\{(1, 5), (2, 5), (3, 5), (1, 7), (2, 7), (3, 7)\}$$

$A \times B$ is called an *ordered pair* and the elements must follow a prescribed order *viz.* the elements of A first and then the elements of B.

A. 1.12. Statements :

Statements are sentences through which we can express some idea and which may be either true or false, but cannot be both. Thus 'Calcutta is in West Bengal' is obviously a statement which is true. Similarly 'seven plus four makes twelve' is also a statement which is false. But consider the statement 'the information you are supplying is false'. In this case if the information is true then the statement is false, but if the information is false then the statement is true. Thus the statement can be both true and false and as such is not a proper 'statement'.

A. 1.13. Open Statement :

When a sentence contains one or more variables such that with definite values prescribed to them the sentence becomes a statement, we call the sentence an 'open statement'. The set of values of the variable which makes the open statement true is called the *solution set* or *truth set*.

Thus the sentence 'A scientist invented the law of gravitation' becomes a 'statement' when we take Newton for the scientist. This is an open statement. Similarly

$$5x+4=14,$$

is an open statement and the corresponding truth set is $\{2\}$.

$$2x+3>9,$$

is an open statement and the truth set consists of all numbers greater than 3. Any open statement using the symbol = is called an equation and that which uses any of the symbols $>$, $<$, or $\neq$ is called an *inequation*.

A. 1.14. Connectives :

In mathematics we are concerned with logical interconnections between 'statements' and also from them build up new statements. For that purpose some symbols are used for making the statements short and precise. Some of them are given below.

The connective $\Rightarrow$: If A, B are two statements such that the statement A implies B, we may write

$$A \Rightarrow B.$$

This may be expressed in any one of the following ways.

(*i*) If A then B.

(*ii*) B if A.

(*iii*) A sufficient condition for the truth of B is the truth of A.

(*iv*) A necessary condition for the truth of A is the truth of B.

Ex. (i) $x=5 \Rightarrow x^3=125$.

Thus $x=5$ implies $x^3=125$; or $x=5$ is a sufficient condition for $x^3=125$.

(ii) PQR is a triangle $\Rightarrow PQ+QR > PR$.

The Connective $\Leftrightarrow$: If A and B are two statements such that

$$A \Rightarrow B \text{ and also } B \Rightarrow A$$

then we may write

$$A \Leftrightarrow B$$

and may say that A is equivalent to B or that A implies B and B implies A. This connective may be expressed in one of the following ways.

(*i*) A if and only if B (A iff B).

(*ii*) A necessary and sufficient condition for the truth of A is the truth of B and vice versa.

(*iii*) A and B are equivalent statements.

Ex. (i) PQRS is a parellelogram $\Leftrightarrow PQ=RS$ and $PS=QR$.

The connectives. $\wedge$, $\vee$:

The symbol $\wedge$ stands for *and* and the symbol $\vee$ stands for *or*. Thus $A \wedge B$ is a statement which is true if A is true and also B is true. Similarly $A \vee B$ is a statement which is true if A is true or B is true or both A and B are true.

Ex. (i) $ab=0$, a, b are real numbers $\Leftrightarrow a=0 \vee b=0$

(ii) $a=5 \wedge b=7 \Rightarrow a+b=12$

(iii) $a^2-3a+2=0 \Leftrightarrow a=2 \vee a=1$

Negation $\sim$:

If A denotes a statement, then $\sim$ A denotes the negation or denial of A.

Ex. (i) If A denotes the statement $a=5$ then $\sim$ A denotes the statement $a \neq 5$
Thus the statement $\sim$ A is true or false according as the statement A is false or true.

The above symbols are sometimes called logical connectives.

A. 1.2. Rules of combination in the Algebra of Sets :

Given two sets we can combine them to form new sets. We are however not restricted to combine only two sets at a time and may deal in a similar way with more than two sets.

The *Union* of two sets A and B, written as

$$A \cup B,$$

is defined to be the set of all elements which are in either A or B or both. Thus we may define

$$A \cup B=\{x : x \in A \text{ or } x \in B\}$$

It is always convenient to have geometric pictures through which one can visualize sets and operations in sets. Thus we can represent the Universal set U by a rectangular area in a plane so that the points within it may represent all the elements of U. Any set may then be represented by an area within this rectangle. In Fig. A.1.1 we have represented the sets U, A and B, where U is the universal set.

All the elements of A are represented by points within the closed area marked A and so on. From this definition we easily have the following results :

(*i*) $A \cup A=A$, (*ii*) $A \cup U=U$, (*iii*) $A \subseteq B \Leftrightarrow A \cup B=B$.

If the set A has elements a_1, a_2, a_3, and the set B has elements b_1, b_2, b_3, then $A \cup B$ has elements $a_1, a_2, a_3, b_1, b_2, b_3$, thus in Fig. A. 1.1 $A \cup B$ is indicated by the shaded area.

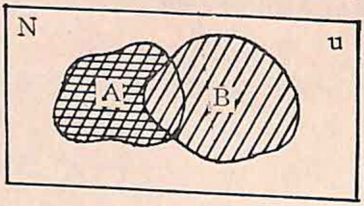

Fig. A. 1.1. The Venn diagram showing the Union of two Sets.

The *intersection* of two sets A and B, written as $A \cap B$, is the set of all elements which are in both A and B. We then have

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

and $A \cap B$ is the common part of the sets A and B. In Fig. A. 1.2.

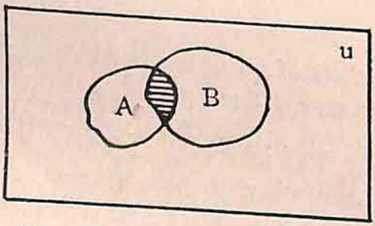

Fig. A. 1.2. The Venn diagram showing Intersection of two Sets.

$A \cap B$ is represented by the shaded area. If therefore $A \cap B$ is a non-empty set then A and B will intersect. Also $A \cap A'$ is a null set. We also have the following results :

(*i*) $A \cap A = A$, (*ii*) $A \cap U = A$, (*iii*) $A \subseteq B \Leftrightarrow A \cap B = A$.

If the set A has elements a_1, a_2, a_3, and the set B has the elements b_1, b_2, b_3, a_2 then $A \cap B$ has only the element a_2.

Two sets are said to be *disjoint* if their intersection is *empty*, that is, a null set. Thus if A is the set of positive integers and B is the set of negative integers then $A \cap B$ is a null set and A and B are disjoint sets.

The *difference* between two sets A and B, denoted by $A-B$, is the set of all elements of A which are not in B. We thus have

$$A-B=A \cap B'.$$

Cor. 1. For any set B, the set A satisfies the condition

$$A=(A \cap B) \cup (A-B).$$

Cor. 2. $B \cap (A-B)$ is a null set.

It is possible to see the validity of the above results by referring to the diagrams similar to Figs. A. 1.1 and A. 1.2.

A. 1.21. Venn diagram:

The terms *set* and *elements* have an intuitive notion described earlier. To strengthen this idea and also provide some justification for the basic laws which are valid in the agebra of sets the concept of Venn diagrams may be introduced. John Venn, a British mathematician of the nineteenth century, evolved a graphical means of illustrating logical truths. In a Venn diagram the set of points interior to a rectangle is taken as the universal set. Any other set which must be within the universal set is represented by points interior to another closed region within the rectangle. By shading appropriate areas all combinations of the sets can be represented graphically. This has been shown in sec. A. 1.2. It may be pointed out that the diagrammatic method mentioned above, does not constitute a proof, but is only an illustration which makes the laws appear plausible. It is however, possible to convince ourselves intuitively about the validity of several results that may be derived by drawing pictures on the Venn diagram.

Ex. 1. Examine intuitively which of the following are always true for arbitrary sets A, B, and C.

(*i*) $A \cup A \cap B = A$; $A \cap (A \cup B) = A$.

(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(*iii*) $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$.

A. 1.22. Fundamental laws in the Algebra of Sets :

The following are some of the basic laws which are valid in the algebra of sets and may be established through intuition or by referring to the Venn diagram. If A, B, and C are arbitrary sets then we have the following results :

(*i*) Commutative laws :

$A \cup B = B \cup A$; $A \cap B = B \cap A$.

(*ii*) Associative laws :

$A \cup (B \cup C) = (A \cup B) \cup C$;

$A \cap (B \cap C) = (A \cap B) \cap C$.

(*iii*) Distributive laws :

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(*iv*) Laws of complementation :

$U' = \phi$; $\phi' = U$;

$A \cup A' = U$; $A \cap A' = \phi$, $(A')' = A$.

ϕ represents a *null* set.

(*v*) Laws of Tautology :

$A \cap A = A$; $A \cup A = A$.

(*vi*) Laws of absorption :

$A \cap (A \cup B) = A$; $A \cup (A \cap B) = A$.

(*vii*) Laws of De Morgan :

$(A \cap B)' = A' \cup B'$; $(A \cup B)' = A' \cap B'$.

It may be pointed out that the laws similar to (*i*), (*ii*) and the first one of (*iii*) stated above are valid also in the algebra of real

numbers. The other laws which are valid in the algebra of sets make its scope much wider. The laws (*vi*) and (*vii*) can be derived by using the definitions of *union* and *intersection* and the laws (*iv*) follows from the definition of the universal set and the null set. We easily have,

$$\begin{aligned} A \cap (A \cup B) &= (A \cap A) \cup (A \cap B) \\ &= (A \cap U) \cup (A \cap B) \\ &= A \cap (U \cup B) \\ &= A \cap U \\ &= A. \end{aligned}$$

Similarly

$$\begin{aligned} A \cup (A \cap B) &= (A \cup A) \cap (A \cup B) \\ &= A \cap (A \cup B) \\ &= A. \end{aligned}$$

We may establish (*vii*) in the following way :

$$\begin{aligned} (A \cap B) \cap (A' \cup B') &= (A \cap B \cap A') \cup (A \cap B \cap B') \\ &= (A \cap A' \cap B) \cup (A \cap B \cap B') \\ &= (\phi \cap B) \cup (A \cap \phi) \\ &= \phi. \end{aligned}$$

Also

$$\begin{aligned} (A \cap B) \cup (A' \cup B') &= (A' \cup B') \cup (A \cap B) \\ &= (A' \cup B' \cup A) \cap (A' \cup B' \cup B) \\ &= (A' \cup A \cup B') \cap (A' \cup B' \cup B) \\ &= (U \cup B') \cap (A' \cup U) \\ &= U \cap U \\ &= U. \end{aligned}$$

Hence by definition of the complement we get,

$$(A \cap B)' = A' \cup B'.$$

Also we have

$$(A' \cup B')' = A \cap B.$$

But if A be the complement of A′ then A′ is the complement of A. Hence from the above we get

$$(A \cup B)' = A' \cap B'.$$

An interesting and useful property evident in the above laws leads to the *principle of duality.* It states that if in an identity each *union* is replaced by *intersection*, each *intersection* by *union*, the *universal* set by the *null* set, and the *null* set by the *universal* set, then the resulting equation is also an identity.

We give below some examples to illustrate the use of the algebra of sets in problems of elementary mathematics.

Ex. 1. In a class of 50 students, 35 like mathematics, 15 like history and 10 like neither. Find how many students like both mathematics and history.

Let M be the set of students who like mathematics, H the set of students who like history, and N the set who like neither. The corresponding Venn diagram is as follows :

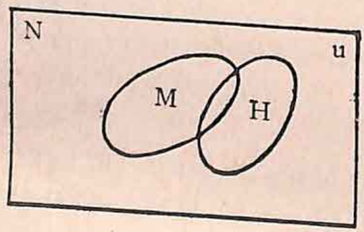

Fig. A.1.3

In the present case therefore,

$$U=50,\ M=35,\ H=15, \text{ and } N=10.$$

The set of students who like both mathematics and history is then $M \cap H$ and from the above Venn diagram we have,

$$M - M \cap H = M \cup H - H$$
$$= N' - H$$

$\therefore \quad 35 - M \cap H = (50 - 10) - 15$

$\therefore \quad M \cap H = 10.$

Hence only 10 students like both mathematics and history.

Ex. 2. A survey shows that 66% of the Indian people like milk whereas only 45% like meat and 20% like both. Show that only 9% of the people like neither milk nor meat.

Ex. 3. Altogether 500 students appeared in an examination in which 300 failed in English, 200 failed in General knowledge, 120 failed in Mathematics, 70 failed in English and Mathematics, 50 failed in English and General knowledge, 20 failed in General knowledge and Mathematices, and 15 failed in English, General knowledge and Mathematics. Find the number of students who failed in subjects other than English, General knowledge, or Mathematics.

Ex. 4. Show that if a line intersects a plane not containing it, then the intersection is a single point.

We know that all lines and planes are sets of points. Through two given points there is exactly one line containing them, and through three given non-collinear points can pass only one plane. It then follows that if two points lie in a plane, then the line containing them also lies in the plane.

If therefore L be a line intersecting a plane E, then we are required to prove that, $L \cap E$ is a point P say. If possible let Q be a second point of intersection, then Q also lies in $L \cap E$. Hence $L = \overleftrightarrow{PQ}$, and as P and Q both lies on E, so $\overleftrightarrow{PQ}$ lies on E, which contradicts the hypothesis. Hence

$$L \cap E$$

cannot contain any point other than P.

Ex. 5. Show that if two lines intersect, then their union lies in exactly one plane.

Let L and L_1 be two lines intersecting at the point P. Then $L \cap L_1$ is the point P. Let Q be a point on L_1 other than

P. Then there is a plane E containing L and Q and hence E is defined by $L \cup L_1$ and no other plane contains $L \cup L_1$.

Ex. 6. Show that if a plane intersects two parallel planes, then it intersects them in two parallel lines.

Ex. 7. If A, B, C, D are arbitrary sets, show that

(*i*) $A \cup (B \cap (C \cup D)) = (A \cup B) \cap (A \cup C \cup D)$.

(*ii*) $(A \cap B') \cup (A \cap C \cap (B \cup D)) = A \cap (B' \cup C)$.

Ex. 8. If X, Y, Z are arbitrary sets, show that

$(X \cap Y) \cup (X \cap Y') \cup (X' \cap Y) \cap (X \cup Y \cup Z \cup (X' \cap Y' \cap Z)) = X \cup Y$.

A. 1.23. Use of Algebra of Sets in the Algebra of Logic :

We have defined earlier that if

$$A \subseteq B,$$

then the set A is a subset of B. The symbol $\subseteq$ without specific reference to any set is defined as *set inclusion*. It then follows from the definition that

$$A \subseteq B \Leftrightarrow A \cap B' = \phi.$$

The following are some important results which follow the definitions given above.

(*i*) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If a be an arbitrary element of A, then since $A \subseteq B$, we have $a \in B$. Also since $B \subseteq C$, we have $a \in C$.

But a is an arbitrary element of A, hence $A \subseteq C$.

(*ii*) If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

If a be an arbitrary element of A, then from the given condition we have

$$a \in B \quad \text{and} \quad a \in C.$$

Hence $a \in B \cap C$ and so $A \subseteq B \cap C$, as a is an arbitrary element of A.

(*iii*) If $A \subseteq B$, then $A \subseteq B \cup C$ for any set C.

Since $B \subseteq B \cup C$, for any set C, the required result follows.

(*iv*) $A \subseteq B$, if and only if $B' \subseteq A'$.

Let b' be an element of B', then b' is not an element of B. But every element of A is an element of B and hence b' is not an element of A and must be an element of A'. Since b' is an element of B' it follows that

$$B' \subseteq A'.$$

Next assume that $B' \subseteq A'$, then from the above we have,

$$(A')' \subseteq (B')' \text{ that is } A \subseteq B.$$

Thus it follows that

$$A \subseteq B \text{ if and only if } B' \subseteq A'.$$

One of the interesting application of the above results is in solving problems of a logical nature. The algebra of sets can very much simplify such problems and help in arriving at precise logical conclusions. The following examples are interesting illustrations.

Ex. 1. Obtain the logical conculsions which follow the statements given below.

(a) A student who does not study hard is not a good student.

(b) Good students receive good grades.

(c) A student who studies hard and receives good grades will secure a fine job.

Let H be the set of students studying hard, G a set of good students, R a set of students who get good grades and F a set of students who secure fine jobs.

Thus by (a) we have,

$$H' \subseteq G' \quad \dots \quad (1)$$

Similarly by (b) we have

$$G \subseteq R \quad \dots \quad (2)$$

and by (c) we have

$$H \cap R \subseteq F \quad \dots \quad (3)$$

Hence from (1) we get $G \subseteq H$ and combining this with (2) we get $G \subseteq R \cap H$. Hence by (3) we get

$$G \subseteq F.$$

Thus the logical conclusion is that 'A good student will secure a fine job'.

Ex. 2. What conclusion can be arrived at from the following statements.

(a) An honorable man never lies. (b) A dishonorable man is never perfect. (c) Perfect men are always tactful. (d) Every tactful man tells an occasional lie.

Let H be the set of honorable men, L the set of liars or set of those who tells an occasional lie, P the set of perfect men and T the set of tactful men.

Hence from the given statements we get :

$$H \subseteq L' ; H' \subseteq P' ; \quad P \subseteq T ; \text{ and } T \subseteq L.$$

We therefore have

$$P \subseteq H,\ H \subseteq L', \quad \text{and hence } P \subseteq L'.$$

$$P \subseteq T,\ T \subseteq L, \quad \text{and hence } P \subseteq L.$$

Hence $P \subseteq L \cap L' = \phi$.

Thus no perfect man is available under the above conditions.

Ex. 3. Given that for certain sets X, Y, Z, V, and W the following conditions hold :

$$X \subseteq Z,\ Y' \subseteq X',\ Z \cap V' = \phi.$$

Show that

$$X \subseteq Y \cap V \cup W.$$

A. 1.3. Boolean Algebra :

Boolean algebra is a part of 'modern algebra' or 'abstract algebra'. The alegbra of sets defined in earlier sections is one of the special types of Boolean Algebra. Its development is entirely *intuitive*. This is used considerably in the design of *computer* circuits. It is also often referred as the 'algebra of

logic'. Any device that has two states or conditions, may be represented by Boolean algebra notations. In consequence it may be used in the analysis of electrical circuits, where there are switches that are either closed or open, and also where current is either flowing or not flowing. Although computer structures are complex still Boolean algebra can simplify them considerably. In consequence Boolean algebra has also developed along with the computers and has proved very useful in the design and trouble-shooting of computers. A knowledge of Boolean algebra is thus valuable also to the technicians.

In the algebra of real numbers we have a given set whose elements are prescribed. We also prescribe there the rules of operations or combinations to be applied to the elements of the set and these define the algebraic structure. The system has some prescribed properties and obey some postulates. We can thus define different forms of algebra in which some operations are prescribed and the rules of their operations on the elements are assumed. Boolean algebra has a structure which we shall define below along with other definitions required for the purpose.

A. 1.31. Definitions and postulates :

Binary operations : An operation (say o) which when applied to two elements of a set A gives a unique element also of the same set, is called a Binary operation. Thus if a, b are the elements of a set then a unique element c, also of the same set, exists such that,

$$c = a \; o \; b.$$

Addition, subtraction, multiplication, and division are obviously Binary operations. In the algebra of sets, union and intersection are also Binary operations. The Binary operations may also satisfy some laws which are familiar in elementary algebra, such as the 'associative' laws, 'commutative' laws etc.

Identity Elements : An element e of a set A is an identity for the binary operation o if and only if,

$$a \; o \; e = e \; o \; a = a,$$

6

for every element a of the set A. In the set of all integers we observe that

$$a+0 = a \text{ and } a \times 1 = a.$$

Thus 0 is an identity for the operation of addition and 1 is an identity for the operation of multiplication.

Idempotent laws : If for all elements a and the operation o we have,

$$a \, o \, a = a$$

then the set is idempotent. In the algebra of real or complex numbers there is no law analogous to this. In the algebra of sets we have observed that,

$$a \cap a = a, \text{ and } a \cup a = a.$$

Hence the set there is idempotent.

Associative Law : A binary operation o on the elements of a set A is associative, if and only if, for every a, b, c, in A,

$$a \, o (b \, o \, c) = (a \, o \, b) o \, c.$$

Commutative law : A binary operation o on the elements of a set A is commutative if and only if for every a and b in A,

$$a \, o \, b = b \, o \, a.$$

Distributive law : Two binary operations o and $*$ say, on the elements of a set A are distributive if and only if, for every elements a, b, c, of A,

$$a \, o \, (b * c) = (a \, o \, b) * (a \, o \, c).$$

We have seen earlier that in the algebra of sets, the operations 'intersection' and 'union' are both associative and commutative but the distributive law for multiplication over addition is different from that of addition over multiplication and we have,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

and

$$A \cup (B \cap C) \quad (A \cup B) \cap (A \cup C).$$

In the algebra of real or complex numbers there is no analogous

law for the distributive property for addition with respect to multiplication, there exists one for multiplication over addition.

With the above definitions for the operations we can following Huntington, define Boolean algebra as follows :

Boolean Algebra : A class of elements A, together with two binary operations o and $*$ is a Boolean Algebra if and only if the following postulates hold :

(*i*) The operations o and $*$ are commutative.

(*ii*) Each of the operations o and $*$ is distributive over the other.

(*iii*) There exists in A distinct identity elements a (say) for the operation $*$ and d (say) for the operation o, so that for all $x \subseteq A$,

$$x * a = x \text{ and } x \, o \, d = x$$

(*iv*) For every x in A there exists an element x' such that

$$x * x' = d, \text{ and } x \circ x' = a.$$

It can be easily shown that the algebra of sets discussed earlier, satisfies all the postulates stated above and hence is a Boolean algebra. Similarly an arbitrary Boolean algebra can be interpreted as an algebra of sets associated with some specially chosen universal set.

Ex. 1. Show that the set A with elements a, b, c, d and operations o and $*$ defined below is a Boolean algebra.

o	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	b
d	d	b	b	d

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

Ex. 2. Show that the class A consisting of 0 and 1 alone (which are also the identity elements relative to the operations

(+) and (.) respectively) together with the operations defined by the following tables, is a Boolean algebra.

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

The above rules of operation apply to the algebra of *Switching circuits* which has a set containing only two elements. The algebra of Switching circuits is thus a Boolean algebra.

In the analysis of the digital computer circuits we are mainly concerned with two operations which correspond to the two 'gates' namely, the AND gate and the OR gate. We are also concerned with two elements comprising the set corresponding to 'pulse' or 'no pulse' or True and False, for the input. If we represent them by the elements 1 and 0 and take (+) for the OR gate and (.) for the AND gate, we observe that the algebra of the digital computer circuit is a Boolean Algebra and we can use the results of this algebra in the analysis of the digital computer circuits.

A. 1.4. Truth Table:

In Boolean algebra if we deal with sets each having only two elements we can derive the nature of the function and also examine their equality numerically. If each set has two elements 1 and 0, then we can obtain the value of a Boolean function by assigning all possible values to the variables which appear in the function, and prepare a table for the value of the function. A table which shows all possibilities of the variables of an equation and their effect on the output or results of the equation is called a *Truth Table.*

A table for two variabels would thus have four states since each variable can take either of the two values 1 and 0, and hence the possible number of combinations are 2×2. Similarly for three variables the number of states will be eight, and so on. Thus corresponding to the equation,

$$A+B+C=F \quad (\text{or} \quad A \cup B \cup C=F)$$

the Truth Table is as follows.

Table 1

OR Truth Table

Row	A	B	C	F
1	0	0	0	0
2	0	0	1	1
3	0	1	0	1
4	0	1	1	1
5	1	0	0	1
6	1	0	1	1
7	1	1	0	1
8	1	1	1	1

Similarly Truth Table for the function,

$$F=(A \cap B' \cap C') \cup (A \cap B' \cap C),$$

is as follows :

Table 2

Row	A	B	C	B′	C′	F
1	0	0	0	1	1	0
2	0	0	1	1	0	0
3	0	1	0	0	1	0
4	0	1	1	0	0	0
5	1	0	0	1	1	1
6	1	0	1	1	0	1
7	1	1	0	0	1	0
8	1	1	1	0	0	0

A. 1.41. Boolean function :

By Boolean function is meant any expression which is a combination of a finite set of symbols, each representing a constant or a variable, associated with the operations of (+), (.) or (′), or their equivalents. Thus

$$(a+b').\,c+a'.b.x$$

is a Boolean function provided the symbols a, b, c, x represent elements of a Boolean algebra.

A Boolean function is said to be in *disjunctive normal form* in n variables $x_1, x_2, x_3, x_4, \ldots, x_n$ for $n > 0$, if the function is a sum of terms of the type,

$$f_1(x_1)f_2(x_2)f_3(x_3)\ldots\ldots f_n(x_n),$$

where $f_i(x_i)$ is x_i or x_i' for each $i=1, 2, 3, \ldots n$ and no two terms are identical. Thus if x, y, z are the variables of a Boolean algebra then,

$$xyz'+xy'z+x'yz$$

is a Boolean function in the disjunctive normal form, (xyz' represents $x.y.z'$).

When a function is represented by its Truth Table, it is completely determined by the value it assumes for each possible assignment of 0 and 1 to the respective variables. This suggests that a function can be constructed in a disjunctive normal form by reference to its Truth Table. For each set of values for the variables which make the function 1, a corresponding term is included in its disjunctive normal form. The sum of such terms gives the function, though that may not be its simplest form. Thus the function represented in Table 2 above is represented by

$$AB'C'+AB'C$$

since the value of F is 1 only in the fifth and the sixth row.

A Boolean function is said to be in *conjunctive normal* form in n variables $x_1, x_2, x_3, x_4, \ldots, x_n$ for $n > 0$, if the function is a product of factors of the type,

$$f_1(x_1)+f_2(x_2)+f_3(x_3)+\ldots\ldots+f_n(x_n),$$

where $f_i(x_i)$ is x_i or x_i' for each $i=1, 2, 3, \ldots, n$, and no two factors are identical.

Thus if x, y, z are the variables in a Boolean algebra then

$$(x+y'+z)(x'+y+z')(x'+y'+z')(x+y+z)$$

is a Boolean function in the *conjunctive normal* form. When a function is defined by its Truth Table its conjunctive normal form can be obtained by noting the value of the variables when the function takes the value 0 in the Truth Table. For each set of variables which makes the function 0, a corresponding term is included in its conjunctive normal form. The product of such terms give the function which however, may not be in its simplest form.

Ex. 1. Two functions are defined by the following Truth Table in which x, y, z are the variables in a Boolean alegbra :

Row	x	y	z	F_1	F_2
1	1	1	1	0	1
2	1	1	0	0	0
3	1	0	1	1	1
4	1	0	0	1	0
5	0	1	1	0	0
6	0	1	0	1	1
7	0	0	1	0	1
8	0	0	0	0	1

Find F_1 and F_2 in the disjunctive as well as the conjunctive normal form.

From the values of the function it follows that

$$F_1=xy'z+xy'z'+x'yz',$$

and

$$F_2 = xyz + xy'z + x'yz' + x'y'z + x'y'z',$$

in the disjunctive normal form. Similarly in the conjunctive normal form

$$F_1 = (x'+y'+z')(x'+y'+z)(x+y'+z')(x+y+z')(x+y+z),$$

and

$$F_2 = (x'+y'+z)(x'+y+z)(x+y'+z').$$

In the design of circuits the Boolean functions are often constructed in the manner discussed above. These forms directly show the nature of the series—parallel circuits that are to be used in representing the function.

APPENDIX II

SOME BASIC CONCEPTS IN TRANSFORMATION GEOMETRY

A. 2. The aim of a course in geometry is not only a training in logic. The traditional course in geometry is based on Euclid in which logic has always taken precedence and the materials have been incorporated only when an abstract logical proof is possible, without regard to whether they are essential or trivial. Moreover emphasis is laid on the 'rigid congruence of triangles'. This has seriously restricted the development of new concepts. According to Felix Klein, geometry should concern also with the study of those properties of figures which are left invarient under all 'mappings' of a group. Also for every group of mappings we can develop a particular geometry. It is therefore desirable to expose the young intelligent pupils to the ideas of 'transformation geometry'. This will give them the power to tackle with confidence varied and intricate problems outside the domain of Euclidean geometry. With that in view a few topics have been incorporated in the following sections which can be presented to the school students at convenient stages. The concept of Reflection, Translation, Rotation, Symmetry and Enlargement has been developed and suitable illustrations have been presented in order to show the efficiency of these concepts in handling geometrical problems.

A. 2.1. Points, Lines, Planes, Segments and Rays :

In formal mathematics we use postulates in which the underlying ideas are self-evident. In discussing the geometry of planes and space we may regard *space* as a *set* S, and the points of space are the elements of this set. The *lines* will then be a collection of subsets of S and similarly another collection of subsets of S will be a plane. Thus 'all lines and planes are sets of points'.

We also observe that, given any two different points, there is exactly one line (preferably straight line) containing them. Thus if the points are P and Q, then the line containing them is denoted by $\overleftrightarrow{PQ}$. The arrowheads indicate that the line extends to infinity in both directions. A *segment* is defined as a portion of a line with definite end points. Thus the segment from P to Q as in the figure below,

Fig. A. 2.1

may be denoted by $\overline{PQ}$. If the line extends to infinity only in one direction with the other end fixed as shown below,

Fig. A. 2.2

then it is called a *ray* and may be denoted by $\overrightarrow{PQ}$. Here P is the end point of the ray $\overrightarrow{PQ}$. It therefore follows that three different non-collinear points define uniquely a *plane*. If the points be P, Q, and R then the plane containing them may be denoted by $\overleftrightarrow{PQR}$ and the plane stretches out to infinity in every directions.

A. 2.2. Betweenness :

If A, B, and C are three points on a line L and if they are situated as follows :

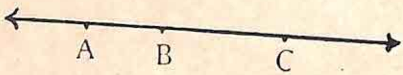

Fig. A. 2.3a

or as,

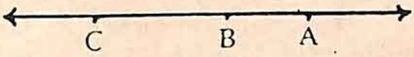

Fig. A. 2.3b

then we define B as between A and C. Thus if for three collinear points A, B, and C we have

$$AB+BC=AC,$$

then B is *between* A and C and we may represent it as $A-B-C$. Thus if $A-B-C$, then $C-B-A$. With this definition the following results follow :

(*i*) If $A-B-C$, then $C-B-A$.

(*ii*) Of any three points in a line exactly one is between the other two.

(*iii*) Any four points of a line can be named in an order A, B, C, D, in such a way that $A-B-C-D$.

(*iv*) If A and B are any two points then there is a point C such that $A-C-B$, and there is a point D such that $A-B-D$.

(*v*) If $A-B-C$, then A, B, and C are three different points of the same line.

A. 2.3. Angles and Triangles :

An angle is formed by two *rays*, having the same end point, as shown below :

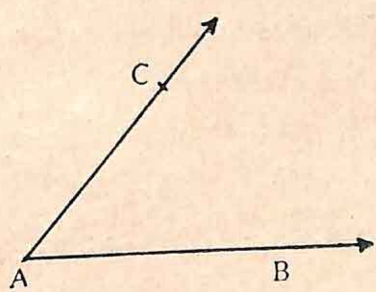

Fig. A. 2.4

Thus an angle is a figure which is the *Union* of two rays with the same end point but not lying on the same line. If the angle is the *Union* of AB and AC, then these rays are called the sides of the angle and the point A is called the vertex. The angle is denoted by the symbol

$$\angle \text{CAB or } \angle \text{BAC}.$$

A *triangle* is formed by joining three non-collinear points. Thus if A, B, and C are three non-collinear points, then the *set*

$$\overline{AB} \cup \overline{BC} \cup \overline{AC},$$

is called a triangle.

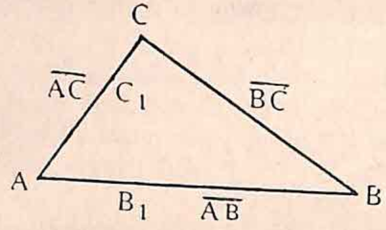

Fig. A. 2.5

The three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called its sides, the points A, B, and C are called its vertices, and the triangle is denoted by the symbol $\triangle ABC$.

The above definitions lead to the following results :

(*i*) If A and B are any two points, then $\overline{AB}=\overline{BA}$.

(*ii*) If C is a point on the ray $\overrightarrow{AB}$, other than A, then

$$\overrightarrow{AB}=\overrightarrow{AC}.$$

(*iii*) If B_1 and C_1 are points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively, other than A, then

$$\angle BAC=\angle B_1AC_1.$$

(*iv*) If $\overline{AB}=\overline{CD}$, then the points A, B are the same as the points C, D, in some order, as the end points of a segment are uniquely determined by the segment.

(*v*) If $\Delta ABC = \Delta DEF$, then the points A, B, and C are the same as the points D, E, and F in some order. Thus the vertices of a triangle are uniquely determined by the triangle.

It is important to note that the symbol = is used in only one sense ; it means "is exactly the same as". Thus when we write $\overline{PQ}=\overline{QP}$, we mean that the sets $\overline{PQ}$ and $\overline{QP}$ have exactly the same elements.

In the above definition of an angle we have not given any idea about its direction or sign. This is sufficient for purpose of Euclidean geometry. But in analytical geometry we require the concept of directed angles in which the initial ray is to be distinguished from the final ray as follows :

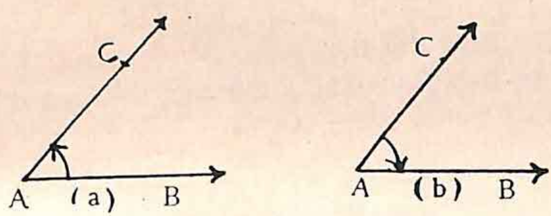

Fig. A. 2.6

An angle defined as above is not a set of points but is an *ordered pair*. Thus (a) is represented by (AB, AC) and (b) is represented by (AC, AB) and they are not equal. One has a sign opposite to that of the other.

A. 2.4 Congruence of segments :

Two figures are congruent if one can be moved so as to coincide with the other. Thus two squares of the same size are always congruent. Similarly two circles of same radius or two

equilateral triangles of the same size are always congruent. Similarly two segments, having the same length, are congruent. We may therefore define as follows :

Two segments $\overline{AB}$ and $\overline{CD}$ are congruent if $AB=CD$. This is then represented as follows :

$$\overline{AB} \cong \overline{CD}.$$

A *Relation* $\sim$, defined on a set A, is called an *Equivalence Relation*, if the following conditions hold :

(*i*) Reflexity : $a \sim a$ for every a.
(*ii*) Symmetry : If $a \sim b$, then $b \sim a$.
(*iii*) Transitivity : If $a \sim b$ and $b \sim c$, then $a \sim c$.

It is easy to show that for segments, congruence is an equivalence relation, since every segment is congruent to itself, if $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$; and finally if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

Ex. 1. Show that if,

(*i*) $A-B-C$, (*ii*) $A'-B'-C'$, (*iii*) $\overline{AB} \cong \overline{A'B}$; (*iv*) $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

Ex. 2. Show that if,

(*i*) $A-B-C$, (*ii*) $A'-B'-C'$, (*iii*) $\overline{AB} \cong \overline{A'B'}$, (*iv*) $\overline{AC} \cong \overline{A'C'}$ then $\overline{BC} \cong \overline{B'C'}$.

Ex. 3. Every segment has exactly one mid point.

A. 2.41. Congruence for Angles :

In the usual functional notation we may write

$$m\angle ABC,$$

to represent the measure of the angle $\angle ABC$. The following results then follow from definition.

(*i*) If $\overrightarrow{AB}$ be a ray on the edge of the half plane H as

shown below, then for every number r between 0 and 180, there is exactly one ray $\overrightarrow{AC}$, where C is on H, such that

$$m\angle CAB = r.$$

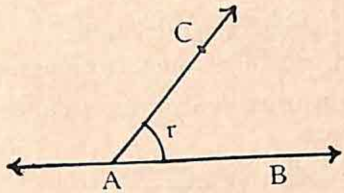

Fig. A. 2.7

(*ii*) If P is the interior of $\angle$ BAC as shown below, then $m\angle BAC = m\angle BAP + m\angle PAC$.

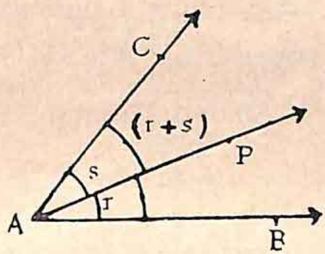

Fig. A. 2.8

(*iii*) If two opposite rays have the same end point and a third ray also has the same end point, then two angles are

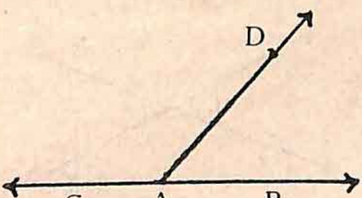

Fig. A. 2.9

formed which form a *linear pair*. Thus in Fig. A. 2.9 the opposite rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ form two angles with the third ray $\overrightarrow{AD}$, and they form a linear pair $\angle DAB$ and $\angle DAC$.

In consequence

$$m\angle BAD + m\angle DAC = 180°$$

and the two angles are called supplementary.

(*iv*) The congruence for angles is defined in terms of their measure. Thus if

$$m\angle ABC = m\angle DEF,$$

then the angles are congruent and we may write

$$\angle ABC \cong \angle DEF.$$

It therefore follows that if the angles in a *linear pair* are congruent then each of them is called a *right angle*.

Two rays are perpendicular if their *union* is a right angle. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular, then we write

$$\overrightarrow{AB} \perp \overrightarrow{AC}.$$

Similarly if the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are perpendicular, we may write,

$$\overleftrightarrow{AB} \perp \overleftrightarrow{AC}.$$

(*v*) Two angles form a *vertical pair* if their sides form pairs of opposite rays as shown in the following figure :

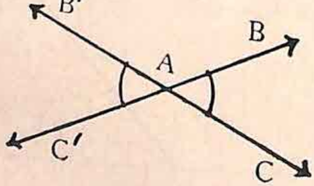

Fig. A. 2.10

Here $\angle BAC$ and $\angle B'AC'$ form a vertical pair. Thus if as in Fig. A. 2.10 $B-A-C'$, $B'-A-C$ and the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AB'}$ are different, then $\angle BAC$ and $\angle B'AC'$ form a vertical pair.

Ex. 1. Show that if two angles form a vertical pair, then they are congruent.

Ex. 2. If two intersecting lines form one right angle, then they form four right angles.

A. 2.5. Congruence and Transformation :

We have so far defined congruence of segments and angles. These definitions may be extended to establish the congruence of triangles and circles. Thus we observe that the two figures are congruent if they have exactly the same shape and size, and one can be moved so as to coincide with the other. Euclid based all his congruence proofs on an assumption, which is certainly reasonable, that "things which coincide with one another are equal". Thus it is apparent that the idea of motion, or superposition was implicit in Euclid's congruence proofs, and he also assumed that the geometric figures can be moved without changing their shape or size. It is however, possible to formulate Euclid's concept by defining the nature of rigid motion or isometry. This leads to the concept of *Transformation geometry*. Some elementary concepts of Transformation geometry may be introduced at suitable stages in order to explain how they can be used in the development of Euclidean geometry. This will also help the students in a better way, to appreciate the modern techniques used in advanced courses on geometry, in which more emphasis is laid on mathematical *structures* and *isomorphisms*. It is possible to develop the whole structure of Euclidean geometry in the plane, on the basis of these new concepts and thereby replacing the traditional methods, but this may not be desirable at the school stage. In the following sections we describe some of the common transformations which may help the students in deriving some geometrical results.

A. 2.51. Reflection on a line :

We consider a straight line p in a plane and place a plane mirror on it so that the plane of the mirror is perpendicular to the plane containing the line p. Each point of the plane will have a corresponding definite point, which we call the *image* of the original point called the *object*. We call this mapping of the plane *onto* itself a *reflection* on the line p, and this may be denoted by the symbol M_p. p is called the axis of reflection. Reflection on a line is thus a transformation which has the following obvious properties and may be assumed for further deductions.

(*i*) To every line p there is a unique reflection M_p, so that if A′ be the image when the object is A, then A will be the image when the object is A′. Thus if

$$M_p(A)=A', \text{ then } M_p(A')=A,$$

and A and A′ will lie on opposite sides of p.

(*ii*) The points of p are fixed and p is a fixed line of the mapping. The line AA′ is also a fixed line.

(*iii*) The reflection M_p leaves distances and angles invarient.

(*iv*) p and AA′ are perpendicular lines, so the line segment joining corresponding points A and A′ is perpendicular to p and is bisected by it. Thus when two different points A and A′ are given arbitrarily, where one is the object and the other is the image, then there exists exactly one reflection M_p such that

$$M_p(A)=A' \text{ and } M_p(A')=A.$$

The axis p is the mediator of the segment $\overline{AA'}$.

(*v*) If two rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ issue from a point P, then there exists only one reflection M_p in which $\overrightarrow{PB}$ will be the image of $\overrightarrow{PA}$ and similarly $\overrightarrow{PA}$ will be the image of $\overrightarrow{PB}$. Then p is the bisector of the angle APB.

The above ideas can be demonstrated through paper folding.

A.2.52. Combination of Transformations:

If a reflection on p maps g into g_1 and then g_1 is mapped into g_{12} by reflection on q, then g_{12} may be defined as obtainable from g by a combined transformation in which the two reflections are performed one after the other. Thus if M_p and M_q represent the two reflections on the lines p and q respectively then the correspondences between a point A and its final image, after the two reflections, A_{12} is given by the relation :

$$A_{12}=M_qM_p\,(A),$$

and M_qM_p defines a new transformation. Thus transformations can be combined.

Cor. 1. If p and q are parallel then M_qM_p represents a translation.

Cor. 2. If a reflection M_p is performed twice we have,

$$M_pM_p=M_p^2,$$

which leaves all points of the plane in the same position. This trivial mapping is called the identity transformation in which all points of the plane continue to occupy the same position, so that every point of the plane is a fixed point.

Cor. 3. M_pM_q is in general a non-commutative transformation as shown in Fig. A. 2.11. It is commutative when p and q are orthogonal lines.

It is evident from Fig. A. 2.12 that when M_p and M_q commutes the transformation M_pM_q is equivalent to a rotation about O, where p and q intersect, which is also the middle point of the segment AA_1', and the rotation is a *half turn.* This transformation is called a reflection in the point O or a half turn about O, and is denoted by H_o. It therefore follows that

$$H_oH_o=H_o^2=M_pM_qM_qM_p=I.$$

These and some other ideas of transformation geometry may be used conveniently in proving the propositions of plane geometry.

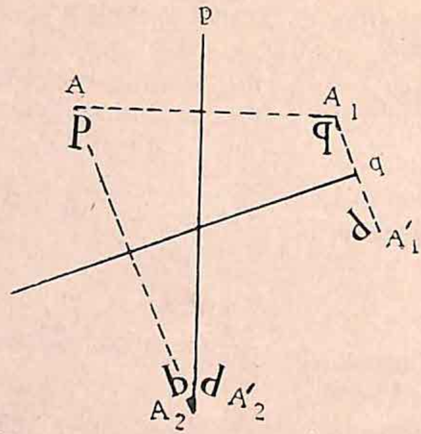

Fig. A. 2.11. Non-commutative combination of Reflections.

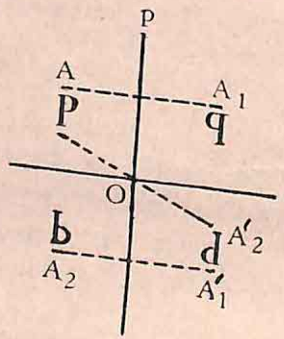

Fig. A. 2.12. Commutative combination of Reflections.

A. 2.53. Linear Symmetry :

If a reflection M_p maps a figure onto itself, then we say that p is an axis of symmetry of the figure. If the figure is mapped onto itself by a half turn H_0 then we say that O is a centre of symmetry.

It is also evident that we can fold each symmetrical figure along the line of symmetry, and when this is done the two halves

of the figure will coincide in all their parts. We thus observe that the process of showing equivalence between geometrical figures through *paper folding* is a direct consequence of the phenomenon of reflection in 'transformation geometry'.

In order to construct a segment symmetrical to a given segment AB with respect to a given line of symmetry p (say), it is only necessary to obtain A′ and B′ as image of A and B obtained by reflection on p. The segment A′B′ is then symmetrical to AB with respect to p. Thus a linear symmetry transforms a segment into a congruent segment and an angle into a congruent angle. Also it can be shown that the point of intersection of symmetrical lines lies on the axis of symmetry, and that the bisector of the angle lies on the line of symmetry of its sides.

Thus reflections are transformations which leave distances and angles unchanged. Hence if a figure is mapped by a product of a finite number of reflections, then its image is congruent to the original. Thus finite product of reflections are *congruences* or *isometries*.

Some symmetrical figures have more than one axis of symmetry. Thus a rectangle has two lines of symmetry, a square has four lines of symmetry. In the case of circles, any line passing through the centre is a line of symmetry.

Ex. 1. Obtain the axis of symmetry of two given points.

Ex. 2. Given the axis of symmetry p, and a point A not belonging to it, obtain the point A′ symmetrical to the point A with respect to the axis p.

Ex. 3. Draw the line of symmetry of a given angle.

Ex. 4. Given two intersecting circles, show that the line passing through their centres is the line of symmetry of points of their intersection.

Ex. 5. How many circles can be drawn through two given points.

Ex. 6. Establish the congruence of triangles by using the symmetry properties or the properties of reflection, in all the following cases :

(*i*) SSS, (*ii*) SAS, (*iii*) ASA.

A. 2.6. Translations :

If we consider two successive reflections on two parallel axes p and q as shown below,

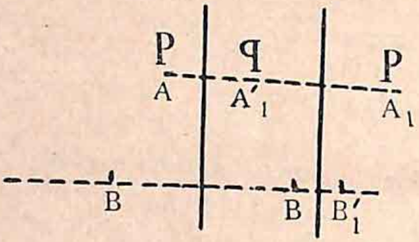

Fig. A. 2.13. Successive Reflections on parallel planes resulting in a Translation.

then it follows that the final image A_1 of A is on the line through A perpendicular to p and q and the distance AA_1 is twice the distance between p and q. Similar will be the case for any other point B on the plane. Thus M_pM_q transforms all points P which has the following properties :

(*i*) The lines joining sets of corresponding points P and P′ are parallel.

(*ii*) All points are moved the same distance in the same direction.

Such a mapping is called a translation. A translation is thus equivalent to two successive reflections on parallel lines. A translation is fixed if one arbitarary point P and its image P′ are given. PP′ defines the magnitude as well as the direction of the translation. If V represents this vector then the magnitude of V is the length of the segment PP′ and the direction of V is parallel to PP′. We then have the following :

Theorem : A translation T with vector V can be represented in infinite ways as a product of two reflections, in which the axes of the two reflections are parallel, are orthogonal to V, and are at a distance apart equal to half the magnitude of V.

Since translations are products of reflections they are isometries and have the following properties :

(*i*) Translations are one-one mappings.

(*ii*) Translations are line-preserving transformations in which the image b' of a line b is always parallel to it (see figure A. 2.14).

(*iii*) Translations are direct isometries, that is, congruences which preserve orientation.

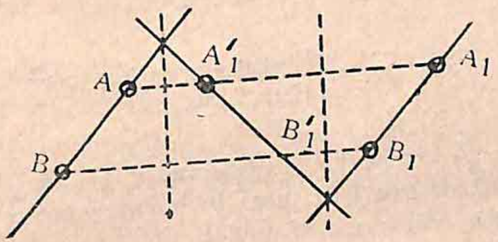

Fig. A. 2.14. Translations lead to Congruences which preserve orientation.

The inverse transformation to a translation T with vector V is also a translation in which the vector is $-V$ and thus has the same magnitude but opposite direction.

A. 2.61. Translations as product of Half Turns :

We have seen that a translation can be considered as the product of two reflections (sec. 2.6). Thus if p and q are parallel and s is perpendicular to both, then a translation T may be represented by the product of two reflections, so that

$$\begin{aligned} T &= M_q M_p \\ &= M_q M_s M_s M_p \\ &= H_B H_A \end{aligned}$$

since the product of two reflections on orthogonal lines is a half turn about their point of intersection.

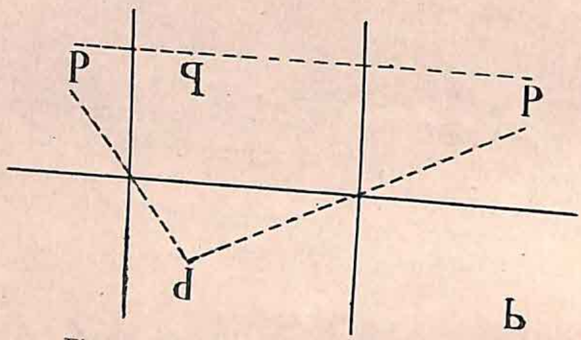

Fig. A. 2.15. Translation equivalent to two Half Turns.

Cor. 1. A translation T with vector V can be represented as the product of two half turns in infinitely many ways. The centres of the half turns A and B are to be so chosen that $AB=\frac{1}{2}V$. The position of A and B is immaterial.

Ex. 1. Show that the product of three half-turns is again a half-turn.

Let

$$\Omega = H_C H_B H_A$$

Since $H_B H_A$ is a translation which has the vector $V=2\overrightarrow{AB}$, if we take the point D such that $\overrightarrow{DC}=\overrightarrow{AB}$, as shown below (Fig. A. 2.16) then obviously,

$$\Omega = H_C(H_B H_A) = H_C(H_C H_D) = H_D.$$
$$= \text{a half turn}$$

Also

$$H_A H_B H_C = H_A(H_A H_D) = H_D = \Omega.$$

Thus in a product of three half-turns the order of the factors may be reversed, so that $H_C H_B H_A = H_A H_B H_C$.

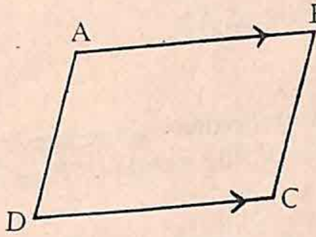

Fig. A. 2.16

Ex. 2. If two opposite sides of a quadrilateral are equal and parallel then the quadrilateral is a parallelogram.

Ex. 3. Show that if

$$H_D H_C H_B H_A = I$$

then A, B, C, D are the vertices of a parallelogram.

We know that $H_B H_A$ is a translation with vector V where $\overrightarrow{AB} = \frac{1}{2}V$. Similarly $H_D H_C$ is also a translation where the corresponding vector is V' say, so that $\overrightarrow{CD} = \frac{1}{2}V'$. Hence if

$$H_D H_C H_B H_A = I,$$

we must have $V = -V'$, that is, $\overrightarrow{AB}$ is parallel, equal, but opposite to $\overrightarrow{CD}$. Hence ABCD is a parallelogram.

Ex. 4. Show that if

$$H_F H_B H_F H_A = I,$$

then F is the mid point of AB.

($H_F H_B H_F H_A = I \Leftrightarrow$ F is the mid point of AB).

$H_F H_A$ represents a translation equivalent to $2\,\overrightarrow{AF}$. $H_F H_B$ represents a translation equivalent to $2\,\overrightarrow{BF}$. If therefore

$$H_F H_B H_F H_A = I,$$

then

$$2\,\overrightarrow{AF} = -2\,\overrightarrow{BF}.$$

Hence F is the mid point of $\overline{AB}$.

A. 2.7. Rotations :

The product of two reflections whose axes intersect at a point, gives a mapping which is called a *rotation*. Thus in Fig. A. 2.17, M_p

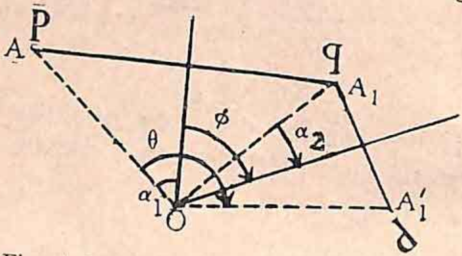

Fig. A. 2.17. Product of two Reflections equivalent to a Rotation.

and M_q are two reflections whose axes intersect at O. It shows that A_1' is the image of A and the mapping is equivalent to a rotation about O. The angle of rotation θ is 2ϕ, where ϕ is the angle between the axes p and q.

The transformation $R = M_q M_p$ is a congruence which preserves orientation and is known as *direct isometries*. The corresponding points A and A_1', where $A_1' = R(A)$, are such that $OA = OA_1'$, and $\angle AOA_1' = \theta = 2\phi$.

A rotation R is fixed when the centre and the angle of rotation is prescribed. The angle of rotation with its magnitude and direction of rotation is called a directed angle, and the direction of rotation is considered positive when it is anticlockwise. We thus observe that if p and q are two lines which intersect at O, then $M_q M_p$ is a rotation about O and the angle of rotation is twice the directed angle between p and q. Similarly a rotation R about a point O and through a directed angle θ, can be expressed in infinitely many ways as the product of two reflections

whose axes p and q can be any pair of lines intersecting at O, and having ϕ, the angel between them, equal to $\frac{1}{2}\theta$.

It therefore follows that rotations are line-preserving transformations and the angle between a line g and its image g' is equal to the angle of rotation.

Ex. 1. Show that a line joining the mid points of two sides of a triangle is parallel to the third side and is also half of the third side.

Let PQR be the triangle in which B is the mid-point of PQ and C that of PR as shown below. Hence we have,

$$H_B H_Q H_B H_P = I, \quad H_C H_R H_C H_P = I.$$

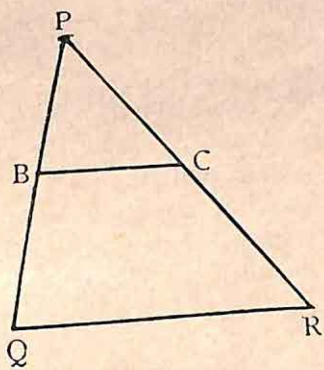

Fig. A. 2.18

$$\therefore \quad H_Q = H_B H_P H_B, \quad H_R = H_C H_P H_C$$

$$\therefore \quad H_Q H_R = H_B H_P H_B H_C H_P H_C$$

$$H_B H_C H_B H_P H_P H_C$$

$$H_B H_C H_B H_C$$

$$(H_B H_C)^2$$

Hence $\overrightarrow{QR} = 2\,\overrightarrow{BC}$, that is, BC is parallel to and half of QR.

Ex. 2. Show that

$$M_f H_B M_f H_A = I \Leftrightarrow f \text{ is the mediator of AB.}$$

Ex. 3. Prove by using reflections and half-turns, that the mediators of the sides of a triangle are concurrent.

A. 2.8. Enlargements :

So far we have considered mappings in which the segments remain equal. It is also possible to consider mappings when the image is an enlarged version of the object.

Let O be a fixed point in the plane and r a non-zero number, positive or negative. If A and A′ are two points on the same straight line through O and are such that

$$\frac{OA'}{OA} = r$$

then A′ is called the image of A and the mapping is called an *enlargement*. O is the *centre* and r the *scale factor* of the enlargement. If $r > 0$, then A and A′ lie on the same side of O, and if $r < 0$, O lies between A and A′.

Thus when O, A and A′ are given we know r and then it is possible to obtain the image of any other point B. If we draw a line through A′ parallel to AB, which may intersect OB

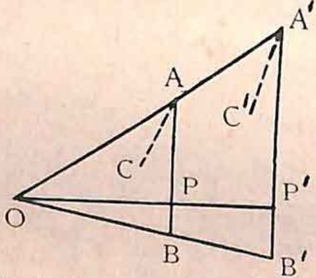

Fig. A. 2.19. Enlargements leading to Line-preserving and Angle-preserving Transformations.

produced at B′, then B′ is the image of B. Conversely if B′ be given we can find B. The construction shows that enlargement is a one-one mapping.

It also follows from Fig. A. 2.19 that if any point P moves on a line AB then its image P′ will trace out a line through A′ parallel to AB. Thus the line A′B′ may be taken as the image of AB. Thus enlargements are line-preserving transformations and any line and its image are always parallel.

If we consider any other line through A, say AC, then its image will be A′C′ which will be parallel to AC. Thus the pair AB, AC is parallel respectively to A′B′, A′C′. Thus the angle between AB and AC is preserved by the transformation. Thus enlargements are angle-preserving transformations. Also since,

$$\frac{OA'}{OA}=\frac{A'B'}{AB}=r,$$

the lengths of corresponding line segments are in the ratio $1 : r$.

Cor. 1. The ratio of the lengths of two image line segments is equal to the ratio of the lengths of the original line segments, so that enlargements preserve ratio of distances.

In a triangle and its image under enlargement the altitudes are also corresponding line segments. Thus areas of a triangle and its image are in the ratio of $1 : r^2$. In general therefore, the areas of a figure and its image under an enlargement are in the ratio $1 : r^2$.

Ex. 1. Construct a line through one point of intersection P of two circles C_1 and C_2, such that the chords on this line have lengths in the ratio $3 : 4$.

Let O_1 be the centre of the circle C_1.

If we join O_1 P and produce it to O_1' such that

$$O_1'P : O_1P = 4 : 3,$$

then the circle C_1' with O_1' as centre and $O_1'P$ as radius is the

image of the circle C_1. If C_1' intersect C_2 at A' then $A'PA$ is the required line which is obtained by joining $A'P$ and producing it to cut C_1 at A.

Since A' is the image of A by construction, we have $PA : PA' = -\frac{3}{4}$, and hence the result.

APPENDIX III

HISTORICAL INFORMATIONS ABOUT SOME EARLY MATHEMATICS AND MATHEMATICIANS

A. 3.1. Some early contributions of Hindu Mathematics :

The historical developments or even backgrounds of a subject, quite often do not find a significant position in its current presentation, but the informations about the eminent scholars of the country and their contribution will no doubt excite interest in the minds of the young pupils of the present generation, who will feel proud about their rich heritage. Some informations about the life and activities of some eminent men will probably help in developing confidence in the minds of the young generation and may encourage them in building their activities. In the following sections we propose to discuss a few topics in mathematics in which Indians in ancient days have made significant contributions long before the scholars of the western countries. We have also given some informations about a few talented Indian Mathematicians, as are available. We have also given some informations about a few internationally reputed mathematicians and scientists whose works in some form may be presented to the students in course of their class works. These informations, it is hoped, will help the students in widening the range of their knowledge and at the same time develop in them a high regard, love and admiration about their motherland.

The discoveries at Mohenjo-daro indicate that even as early as 3,000 B.C. the Hindus lived a highly organised life. The Vedas were composed during this period or probably earlier, which certainly show the high state of civilization prevalent in the country during that period. Subsequently round about 2,000 B.C. the Brahmana literature were developed and in addition to philosophy they also show the attempts of Indians in developing mathematics and science and particularly arithmetic,

geometry, algebra and astronomy. In the following periods, even in spite of several foreign invasions and internal conflicts, the intellectual progress of the country continued. The period 400 B.C. to 400 A.D. was a period of great activity when scholarly works were produced in the country both in arts and sciences. The great astronomical work entitled Surya Siddhanta, which contains considerable amount of well developed mathematical principles was written during this period. The ancient Hindus took Ganita to mean "the science of calculation" and it included astronomy (jyotisa), geometry (ksetra-ganita), 'rule of three, (rasi), fractions (kalasavarna), simple equations (yâvat tâvat), quadratic equations (varga), cubic equations (ghana), biquadratic equations (varga-varga) and permutations and com-binations (vikalpa).

The calculations involved in Ganita were made on a board (pati) with a piece of chalk and hence the name patiganita was adopted. Later in about 628 A.D. Brahmagupta used Bija-ganita as a separate section of Ganita dealing with algebra and named it as Kuttaka in his treatise named Brahma-sphuta Siddhanta. Sridharacharya later (750 A.D.) regarded Pati-ganita and Bij-ganita as separate subjects and since then this distinction has con-tinued to exist. The Hindus used geometry in problems connected with mensuration. The ancient Sulvasutras (800 B.C.) show that the Hindus applied geometry to the construction of alters and in doing so made use of what we know to-day as Pythagorean relation. The rules furnish instructions for finding a square equal to the sum or difference of two given squares and of a square equal to a given rectangle. There also exists the solution of the circle-squaring problem in which the diameter d of the circle and S, the side of the square were connected by the relations,

$$d = (2+\sqrt{2})S/3, \qquad S = 13d/15.$$

The decimal system with base ten has formed the basis of numeration in India even from the days of Rigveda. While the Greeks had no terminology for denominations above the myriad (10^4) and the Romans above the mille (10^3), the ancient Hindus

freely used as many as eighteen denominations. The most important contribution of the ancient Hindus is the *decimal place-value notation* and the *discovery of zero*, and some interesting details about them are given below.

A. 3.11. The decimal place-value system:

From the years 500 to 1400 A.D. there was no mathematician of note in the whole Christian world, but the Hindus and the Arabs made significant contributions also during that period. The Hindus applied the principle of place-value to base ten. There is absolutely no trace of any large use of any other base of numeration in the whole of Sanskrit literature. Reference to the place-value notation appears also in Puranas. The Agni-Purana says : "In case of multiples from the units' place, the value of each place is ten times the value of the preceding place". The Agni-Purana was written not later than second century A.D. It is generally assumed that numerical symbols were invented after writing had been in use for some time and that in the early stages the numbers were written out in full in words. Some historians were of the opinion that writing of numbers were introduced into India following the practices in the West sometimes during eighth century B.C. They assumed that the ancient Indian script, as found in the inscriptions of Asoka were derived from the more ancient writings discovered in Egypt and Mesopotamia.

Recent discoveries have however, rejected all such theories claiming that the Indian script was derived from foreign sources. Pottery belonging to the Megalithic (1500 B.C.) and Neolithic (6000 B.C. — 3000 B.C.) ages, preserved in the Madras Museum have been found to be inscribed with writings and five of these marks are identical with the Brahmi characters of the time of Asoka. The excavations at Mohenjo-daro and Harappa have also produced evidences to show that the ancient Hindus must have possessed a well developed system of numerical symbols.

The Brahmi inscriptions are found distributed all over India and probably the Brahmi script was the national script of ancient

Hindus, and it reached perfection near about 1000 B.C. or earlier. The Brahmi numerals are also an Indian invention. The earliest epigraphic instance in India, of the use of the modern system of notation, which probably was long after their invention, is 594 A.D. No other country in the world has an earlier instance of its use. The Babylonians introduced place-value while using base sixty. The Greeks and Europeans used this system in all mathematical and astronomical calculations until the sixteenth century and it still survives in the division of angles and hours into 60 minutes and 60 seconds. This alone is sufficient to assign a Hindu origin to the numeral forms for the numbers 1 to 9 and the decimal place-value system. Unfortunately it is not known who was the inventor of the new system or where was the exact seat of learning where it was used first.

A. 3.12. The use of zero symbol :

The zero symbol was used by Pingala before 200 B.C. In the Panch-Siddhantika (505 A.D.) zero has been mentioned in several places. The writings of Jinabhadra Gani in the sixth century A.D. offer a conclusive evidence of the use of zero as a distinct numerical symbol used in arithmetical calculations. Bhaskara I has used the place-value numerals with zero to denote the notational places.

These records show why the Hindu-Arabic numeral system is named after the Hindus who invented it and after the Arabs, who transmitted it to western Europe. The present number symbols are found on some stone columns erected in India about 250 B.C. by King Asoka.

In all possibility these numeral symbols were carried by travellers along the Mediteranean coast. They were possibly introduced into Spain by the Arabs who invaded that country in 711 A.D. There were some variations of the number symbols before they stabilized to their present form. The word zero probably comes from the Latin word Zephirum which is the equivalent of Arabic 'sifr', which again is a translation of the Hindu 'sunya' meaning void or empty.

In the following sections we shall present in short some details of a few Hindu Mathematicians who by their works brought prestige for India in the world of Mathematics. We have also given a brief account of a few eminent mathematicians and astronomers whose works may be presented to the school students in some form or other. Their personal life and works will, it is hoped, create interest and bring confidence in the minds of the young pupils who may have the opportunity to know about them.

A. 3.2. Some informations about life and works of ancient Hindu Mathematicians :

Aryabhatta I : Aryabhatta I is considered as one of the pioneers who developed Astronomy in India. Born in 476 A.D. at Kusumpur (near present Patna city) he developed several principles of Algebra which was then designated as 'Kuttaka'. His principal work is known as Aryabhatta Tantra and was published in four volumes containing Arithmetic, Algebra and Astronomy. He indicated probably for the first time, that the earth has a diurnal rotation and also revolves round the sun once in a year, which was later established in Europe by Nicolas Copernicus about a thousand years later.

Aryabhatta discussed the solution of linear equations and general equations of the type

$$\Sigma x - x_1 = a_1,\ \Sigma x - x_2 = a_2,\ \dots,\ \Sigma x - x_n = a_n,$$

where

$$\Sigma x = x_1 + x_2 + x_3 + \dots + x_n.$$

He obtained the formula giving the sum of n terms in arithmetical progression and expressed the number of terms n, in terms of the sum of the series S, first term a and the common difference b, in the following form :

$$n = \frac{1}{2b}\{(8bs + (2a-b)^2)^{\frac{1}{2}} + b\}.$$

This shows that he also discussed the methods of solving quadratic

equations. He was probably the earliest Hindu algebraist to give a treatment of the *indeterminate equations.*

Brahmagupta I : Brahmagupta I was one of the famous astronomers and algebraists of India. Born in 598 A.D. somewhere near Multan province he published Brahmasphuta Sidhanta. Algebra developed in Europe was based on the works of the Arabs, but the Arabs took them from the works of Brahmagupta I. He also made some significant contributions in Geometry. He gave the formula for the area of a triangle in terms of the three sides and also extended that to express the area of a cyclic quadrilateral in the form

$$[(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{2}}.$$

where s is the semiperimeter and a, b, c, d are the sides of the quadrilateral. A remarkable contribution of Brahmagupta was to show that the diagonals p and q of a cyclic quarilateral having consecutive sides a, b, c, d are given by the relations :

$$p^2 = (ab+cd)\ (ac+bd)/(ad+bc),$$

$$q^2=(ac+bd)\ (ad+bc)/(ab+cd).$$

Brahmagupta discussed the solution of the indeterminate quadratic equation of the form :

$$\mathrm{N}x^2 \pm c = y^2,$$

which the Hindus designated as Varga-prakriti, meaning "Square-nature". Brahmagupta showed that if $x=\alpha$, $y=\beta$, be a solution of the equation

$$\mathrm{N}x^2+c=y^2,$$

and $x=\alpha'$, $y=\beta'$, be a solution of the equation

$$\mathrm{N}x^2+c'=y^2,$$

then

$$x=\alpha\beta' \pm \alpha'\beta\ ;\quad y=\beta\beta' \pm \mathrm{N}\alpha\alpha'$$

is a solution of the equation

$$\mathrm{N}x^2+cc'=y^2.$$

This is known as Brahmagupta's lemma.

It follows from the above that if $x=\alpha$, $y=\beta$, be a solution of the equation

$$Nx^2+c = y^2,$$

then $x=2\alpha\beta$, $y=\beta^2+N\alpha^2$, satisfies the equation

$$Nx^2+c^2=y^2.$$

This is known as Brahmagupta's Corollary. The above results show that when only one set of solution of the equation

$$Nx^2+1=y^2,$$

is known, it is possible to obtain an infinite number of solution of the same equation.

Brahmagupta's Lemmas were rediscovered and recognised as important by Euler in 1764 and by Lagrange in 1768. Some historians have incorrectly claimed that Fermát in the seventeenth century first asserted that the equation

$$Nx^2+1 = y^2,$$

where N is a non-square integer, has an unlimited number of solutions in integers.

Brahmagupta also gave specific rules for solving quadratic equations, which is similar to Sridhara's rule, and simultaneous quadratic equations.

Sridhara : Sridhara Acharya was born in 750 A.D. His father Baladeva Sharma was born in Rarh Desh on the west coast of the river Ganges. His book 'Trisalika' is on Arithmetic and was composed on hymns. The treatises of Sridhara contains some limited application of the 'Rule of False position' in problems of arithmetic. Sridhara's rule in finding the roots of a quadratic equation goes by his name even in the books on algebra of the present time. He also discussed the geometrical properties of cyclic quadrilaterals. Sridhara regarded Patiganit and Bijaganit as separate and this has continued till to-day. Sridhara and Brahmagupta both adopted a simple method of squaring a number which is a consequence of the relation

$$m^2 = (m-a)\ (m+a)+a^2.$$

Similarly for obtaining the cube of a number Sridhara used the formula

$$n^3 = \sum_{1}^{n} \{3r(r-1)+1\}.$$

He also discussed methods of obtaining square roots and cube roots of numbers. The principle of "Rule of three" was designated by Brahmagupta, Sridhara and others as 'Trairasika' and different rules were prescribed for solving problems by using this principle. He also discussed 'compound proportion' and the functions of zero in all arithmetic processes.

Bhaskaracharya : Bhaskaracharya was one of the eminent astronomers and mathematicians of India. He was born in 1114 A.D. in a village named Biddarvir in South India. His father was Mahes Acharya. At the young age of 36 he compiled his works under the title of "Siddhanta Siromani" and in four volumes *viz.* (i) Lilavati (Arithmetic), (ii) Bijaganit, (iii) Grahaganitadhyay, and (iv) Goladhyay. The last two dealt with the problems of Astronomy. There is a story connecting the reasons why his book on arithmetic was named after her daughter Lilavati. From astrological calculations Bhaskara found that his daughter was destined to be a widow at a very early age. In order to counteract this misfortune he found out a certain date and time from astrological calculations, when Lilavati should marry in order to avoid what was ordained for her. The precise estimates of time in those days were made through 'water clock', and as Lilavati was watching the sinking water level a small piece of material from her crown fell into the cup and closed the hole, thus stopping the process of time reckoning. The lucky moment thus passed away and Bhaskara failed to alter the fate of her daughter. To console the unhappy girl Bhaskara named his first book after her name, and much of the Hindu arithmetic available to-day are from Lilavati.

Bhaskara gave several approximations to the value of π. He took 3927/1250 as an accurate value, and 22/7 as an approximate

value for π. Aryabhatta, however, in 530 A.D. gave for it the value 62,832/20,000 which comes to 3·1416. Bhaskara gave two remarkable identities.

$$(a \pm b^{\frac{1}{2}})^{\frac{1}{2}} = ((a+(a^2-b)^{\frac{1}{2}})/2)^{\frac{1}{2}} \pm ((a-(a^2-b)^{\frac{1}{2}})/2)^{\frac{1}{2}},$$

which are employed in algebra even to-day for obtaining the square root of a binomial surd. The principle of "rule of three" was put forward by Brahmagupta and Bhaskara and for centuries this was used by the merchants mechanically and its connection with 'proportion' was realised only by the end of the fourteenth century. Even the early European writers on arithmetic used the rule mechanically for a long time. Bhaskara also gave several rules for the solution of 'indeterminate equations'. The idea of zero as an infinitesimal is more in evidence in the works of Bhaskara. He actually used quantities which ultimately tend to zero, and successfully evaluated the differential coefficient of certain functions. He knew that

$$\frac{a}{0} \to \infty \text{ and } \infty + b = \infty,$$

and compared ∞ with the infinite and immutable God.

A. 3.3. Some informations about life and works of western mathematicians :

Pythagoras : Pythagoras was probably a native of Samos belonging to the Ionian colony of Greeks planted on the western shores and islands of what we now call Asia Minor. He lived from about 584 B.C. to 495 B.C. In 529 B.C. he settled at Crotona, a town of the Dorian colony in south Italy and there began to teach philosophy and mathematics. On the advice of his teacher Thales, Pythagoras visited Egypt and gained good experience. He married Theano, the young daughter of his host Milo. The pupils of Pythagoras formed a society which was known as the Order of the Pythagoreans and they exercised great influence far across the Grecian world. As it was the generous practice among members of the brotherhood to attribute all credit

for each new discovery to Pythagoras, it is difficult to ascertain the authorship of each theorem. Pythagoras learnt his mensuration from the priests of Egypt. He was interested in observing different geometrical properties from the arrays of coloured squares on the floors of the temples. He observed there patterns in which a larger square was enclosing another square having exactly half its size. This probably led to the discovery of the great theorem which goes by his name ($a^2+b^2 = c^2$).

Pythagoras was also interested in the more abstract natural objects. He discovered the 'harmonic progressions' in the notes of the musical scale by finding the relation between the length of a string and the pitch of its vibrating note.

Through extensive travel in Egypt and India he absorbed much of mathematics and mysticism.

Euclid : The scene of mathematical learning shifted from Europe to Africa towards the end of the fourth century B.C. Alexander of Macedonia conquered the Grecian world and conceived the idea of forming a great empire, but he died only two years after founding the city of Alexandria. Ptolemy, the successor of Alexander in his African dominions, established a University and took Euclid as one of the earliest teachers. Euclid probably passed his years of tuition at Athens before accepting the invitation of Ptolemy to settle in Alexandria. During his stay at the University he wrote his Elements and other works of importance. Euclid was modest and very fair and conspicuously kind and patient. Some one who had begun to read geometry with Euclid, on learning the first theorem asked, "What shall I get by learning these things ?" Euclid called his slave and said, "Give him three pence, since he must make gain out of what he learns".

In the Elements, Euclid set about writing an exhaustive account of mathematics. The work consisted of thirteen books. Books I, II, IV, V and VI are on lines, areas, and simple regular plane figures, while Book III is on circle. Books VII, VIII and IX give an interesting account of the theory of numbers. G.C.M. and L.C.M. of numbers and the theory of geometrical progression

and the law of indices were also introduced. Book X contains the idea of irrational numbers particularly of the type

$$(a^{\frac{1}{2}} \pm b^{\frac{1}{2}})^{\frac{1}{2}},$$

where a, and b are positive integers. Book XI deals with solid geometry.

The Parallel Postulate of Euclid was probably criticised very much, and hundreds of attempts were vainly made to remove this postulate by proving its equivalent but it was not possible. The vindication of Euclid came with the discovery of non-Euclidean geometry in the nineteenth century, when fundamental reasons were found for some such postulates. Euclid had also some works in Astronomy, music and optics.

Kepler : The German mathematician and astronomer Johann Kepler was born near Stuttgart in 1571. He was educated at the University of Tübingen, where he initially studied Theology and Religion. His love for mathematics and the inner urge to understand the problems of astronomy, later diverted him towards mathematics. In 1594, while he was quite young, he joined the University of Grätz in Austria as a Lecturer in Mathematics and later in 1599 became an assistant to the famous Danish astronomer Tycho Brahe, who was the court astronomer to Kaiser Rudolph II, and was then working at Prague, collecting astronomical data regarding the motion of the planets. After the death of Tycho Brahe in 1601, Kepler was appointed in his master's position and thus had free access to the large mass of data collected by his master. Purely through a process of trial and error and naturally after several unsuccessful attempts, he finally discovered empirically the laws of planetary motion which fitted in, the mass of astronomical data complied by Tycho Brahe. The very simple and elegant laws are the following :

(*i*) All planets move around the sun in elliptical orbits with the sun at one of its focus.

(*ii*) The line joining the sun and a planet sweeps out equal areas in equal intervals of time.

(*iii*) The square of the period of rotation of a planet is proportional to the cube of the semi-major axis of its orbit.

These laws in later years were considered as the basic concepts on which the modern celestial mechanics was built up. The Hindu astronomers discussed considerably the motion of planets as early as the fourth century A.D. and a good record of that is available in Surya Siddhanta, but possibly the laws were not laid down in precise terms. Unfortunately many records of the works of Hindu astronomers and mathematicians are missing today. The ancient scholars of this country were very modest and would prefer to remain anonymous.

Kepler observed that the increment of a function becomes vanishingly small in the neighbourhood of an ordinary maximum or minimum value and Fermát later translated this fact into a process for determining maximum or minimum of a function which we now adopt in calculus.

Kepler used some crude methods of integral calculus in establishing his second law and also extended these methods in the calculations of the volumes of solid bodies, long before calculus was invented by Newton and Leibniz in the seventeenth century A.D. Kepler's works were full of speculation which in several cases were found to be true.

He was not quite happy in his personal life. His favourite child died of small pox, his wife turned mad and died, and he himself was accused of heterodoxy. He suffered from financial difficulties as his salary was quite often in arrears, and thus had to supplement his income by preparing horoscopes. It is reported that he died in 1630 while on a journey to obtain some of his long overdue salary.

Newton : Isaac Newton, the renowned British Mathematician, was born in Woolsthorpe on the Christmas Day of 1642. His father, who was a farmer, died before Isaac was born. Normally he would follow his father's profession, but young Isaac showed great skill and interest in building mechanical devices and conducting experiments. At the age of 18 he entered Trinity

College, Cambridge, and became interested in mathematics while reading a book on astrology which he purchased in a fair. He then read some of the works of Euclid and Descarte's and got an insight into the problems of geometry. He also read the works of Kepler and tried to rationalize the empirical relations deduced by him. At the early age of 23 he created his method of fluxions, which is known to-day as Differential Calculus. At about the same time he performed his first experiment in optics, and formulated the basic principles of the theory of gravitation. In 1669 he succeeded Barrow as the Lucasian Professor at Cambridge. His early works on Optics were published in the Proceedings of the Royal Society. Unfortunately these were criticised by several scientists and Newton was so unhappy about it that he decided not to publish his works further, and some very important of his works remained unpublished until many years after their discovery. In 1675 he presented his 'Corpuscular theory of light' to the Royal Society. In 1679 he verified his 'Law of gravitation' and established the compatibility of his law of gravitation with Kepler's laws of planetary motion which however, was given out by him in 1684. He completed his first volume of the Principia in 1685, and the second volume was completed a year later. The complete treatise in three volumes was published in 1687 under the title 'Philosophiae naturalis principia mathematica'.

In 1689 Newton represented the University in parliament. In 1696, he was appointed Warden of the Mint, and in 1699 he became the Master of the Mint. In 1703, he was elected President of the Royal Society, a position to which he was annually re-elected until his death in 1727 at the age of 84.

Newton's most important work is his Principia in which he presented a complete formulation relating to the 'Laws of motion' and their application to terrestrial and celestial objects. He is probably the greatest mathematician the world has yet produced. His accomplishments were poetically expressed by Pope with the following words :

"Nature and Nature's laws lay hid in night
God said, 'Let Newton be', and all was light"

Newton was however, extremely modest as will appear from his following statement :

'I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then, finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me'. In appreciation of his predecessors he remarked once that if he had seen farther than other men, it was only because he had stood on the shoulders of giants.

Copernicus : Like the Hindu mathematicians at the beginning of the Christian era whose main interest was in Astronomy, even in Europe till the sixteenth century A.D. the mathematicians were mostly astronomers. Astronomical observations and their analysis considerably stimulated mathematics and Nicolas Copernicus of Poland is one of those eminent mathematicians. He was born in 1473 and was educated at the University of Cracow, where he studied mathematics and science. He then went to Bologne to study astronomy under Novara, a foremost Pythagorean. In 1512 he assumed the position of canon of the cathedral of Frauenburg in East Prussia, his duties being that of a steward of church properties and justice of the peace. During the remaining thirty-one years of his life he spent much time in a little tower on the wall of the cathedral closely observing the planets with naked eyes and making untold measurements with crude instruments. After years of observation and analysis Copernicus established his heliocentric hypothesis. He however, assumed that the moon and planets move on a circular path whose centre moves on another circle. He also assumed that each body moves along its circle at a constant speed, though apparently it was not so. A change in speed, Copernicus reasoned would be caused only by a change in motive power, and since God, the cause of the motion, was constant, the effect could not be otherwise. Though the heliocentric hypothesis was correct, his assumptions about the 'circular orbits' and 'constant speed' were wrong and was later corrected by Kepler.

Copernicus hesitated to present his hypothesis, but after

about ten years, only at the request of his friend Rheticus he agreed to publish it. Unfortunately Copernicus received a copy of his book while he was lying paralyzed from an apoplectic stroke from which he never recovered. He died in 1543.

The influence of Geometry was so strong in those days that on the title page of his book entitled 'On the revolution of the heavenly spheres' appeared the legend originally inscribed on the entrance to Plato's academy : "Let no one ignorant of geometry enter here".

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Teacher's Guide to Stage 2

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